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EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.

AND

PROF. E. T. WHITTAKER, M.A., F.R.S.

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The Mathematical Association.

THE ANNUAL MEETING will be held at the LONDON DAY TRAINING COLLEGE, Southampton Row, London, W.C. 1, at 4 p.m., on *Monday, 4th January*, 1926 (Advanced Section); at 10 a.m. and 2.30 p.m., on *Tuesday, 5th January*, 1926 (Ordinary Meeting).

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VOL. XII. DECEMBER, 1925. No. 179.

HOW IS e TO BE INTRODUCED INTO OUR TEACHING?*

BY J. KATZ, B.A.

As soon as the Calculus had descended from its academic heights and had been persuaded to frequent the schoolroom, there was bound to be trouble about e.

The traditional approach to e is through the exponential theorem as we find it presented in a text-book of the so-called "Higher Algebra." But such an approach was clearly out of the question. Whether even students of Higher Algebra, after having worked through fifty examples on the Binomial Theorem, should suddenly be asked to prove that if $f(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+\dots$, then

$$f(x)\times f(y)=f(x+y),$$

is a question of method which I do not propose to discuss here, but I presume everyone agrees that if the Calculus is to be taught to boys in their matriculation year, and (what is more important) if it is to be taught to Polytechnic students who are frequently hazy about the factors of (a^2-b^2), then the question, "How the teacher is to introduce e" requires to be answered. And a really satisfactory answer is not easy to find. Perhaps in trying to teach logarithmic and exponential functions to pupils of tender years or of narrow attainments, we have set ourselves a pedagogic problem to which there is no satisfactory solution. Only experiment and experience can decide.

The various methods of presenting e are admirably summarised in a pamphlet † of the Board of Education issued in 1912. One or other of these methods has been adopted in most of the recent text-books introducing the Calculus to schools. I shall pass one or two criticisms on them before giving the method I have adopted in my own teaching. I have tried it with a class of Polytechnic students, composed principally of young electrical engineers, most of whom had taken Mathematics for two or three years before starting the Calculus with me; with a fifth form that sat for Matric. at the end of the course, and with a sixth form reading for Inter. or Higher Schools.

I must confess that I am far from being satisfied with my own method. A really adequate method must, I think, satisfy each of the following tests:

(1) *The Test of Continuity.* The new matter presented to the pupil must be a natural and inevitable development from what he knows already. I do not

* A lecture delivered before the London Branch of the Mathematical Association, Dec. 1924.

† *The Calculus as a School Subject*, by Mr. C. S. Jackson.

expect even a gifted pupil to discover e for himself, but neither do I want him to gulp down in tabloid form the concentrated essence of what historically has been a slow growing insight.

(2) *The Test of Imaginative Breadth.* e plays a great rôle in analysis—it is more important even than π. Only very few of our pupils will continue their studies to the point that they are able to appreciate what that rôle is. But it is possible to suggest where it is impossible to inculcate. Even the unlearned can be provoked to admiration.

(3) *The Test of Simplicity.* Can the pupil follow? All teaching is, of course, useless if the demonstrations are so involved or so difficult that the pupil cannot follow. But the importance of "simplicity" is exaggerated if the text-book writer's sole object is to get the book-work over in order that he may start the pupil working endless examples.

That too many sacrifices have been made to the desire of "getting on" quickly is the criticism I should make of those text-books which launch the pupil into this branch of the Calculus by asking him, first, to graph $y=\log_{10}x$, then to show from first principles that the gradient when $x=1$ is $\cdot 4343$, and finally, by an ingenious manipulation of proportions, to attain the result

$$\frac{\text{gradient at }(x, y)}{\text{gradient at }(1, 0)}=\frac{1}{x}.$$

I have no doubt that the pupil can follow all this, but he is being led, or rather hustled, to a point which the teacher himself has reached by quite a different route. What this method of rather dainty artifice does not explain, is why the pupil should be made to start on this journey at all, nor does it suggest that there is anything very striking to see at the journey's end.

Except for the absence of "dodges," the same criticism can, I think, be levelled at those text-books which commence this part of the Calculus by graphing $y=a^x$ for different values of a, e^x being defined as the graph whose subtangent in the limit approaches unity.

Of course, all the text-books that begin in this fashion make a more or less perfunctory reference to "growth at compound interest rate" as being somehow connected with the idea of e. That this idea is pushed into the foreground, and very fully discussed, is a distinct merit of that very amusing and provoking book, *The Calculus Made Easy*, by the late Silvanus Thompson. I have no objection to his employing the Binomial Theorem to expand $(1+1/n)^n$ without troubling himself about the remainder. "Sketchy" methods are quite justifiable in certain cases. But what, as I think, is too abrupt a transition occurs in the next step he takes: he proceeds to expand $(1+1/n)^{nx}$. The transition to this step, as I know from experience, puzzles the student. Of course, Thompson wants to establish the striking result that if the expansion of e^x is differentiated term by term, the series recurs. I believe that by Thompson's methods the student gets an insight into the nature of e as a limit, but not into the nature of the function e^x.

I will conclude these criticisms by referring to Professor Nunn's treatment of logarithmic and exponential functions in his work on *The Teaching of Algebra.* His treatment more than satisfies two of the canons of method to which I have referred: for continuity and naturalness of development and for imaginative breadth, his treatment leaves little to be desired. But if I were to follow him at all closely, I should have to contend with the practical difficulty that all my pupils have learnt their common logarithms several years before they touch the Calculus, and they have not learnt them by Professor Nunn's method. Naturally, I revise common logarithms when starting on this part of the Calculus; still, the fact remains that the work in logarithmic functions is more obviously correlated with the work already done in the Calculus, viz. with the differentiation and integration of x^n, than it is with the very early work in common logarithms.

The ideas underlying my own method are derived (1) from Mr. J. M. Child's illuminating edition of *The Geometrical Lectures of Isaac Barrow* (see especially note on pp. 183-186); (2) Tropfke's *Geschichte der Elementar-Mathematik*, vol. ii., especially pages 181-183.

As every one knows, the earliest successes in the Calculus were scored in the field of integration. Wallis and others had succeeded in integrating directly (with no previous knowledge of differentiation) what in modern notation we should write in the form:

$$\int x^n dx = \frac{x^{n+1}}{n+1}.$$

Now there is one value of n for which the formula breaks down, viz. when $n=-1$, *i.e.* when we are trying to effect the quadrature of the equilateral hyperbola $xy=1$.

The many attempts that were made to effect this quadrature led the Jesuit father, Gregory of St. Vincent, to publish in 1647 the beautiful theorem that if the asymptotes of an equilateral hyperbola are taken as axes of x and y, and the abscissae of successive points on the curves are in geometrical progression, then the areas enclosed by the first ordinate, the successive ordinates, the curve, and the asymptote will be in arithmetical progression. (Gregory of St. Vincent, as a matter of fact, stated the theorem in a more general form in which it is true for any hyperbola.)

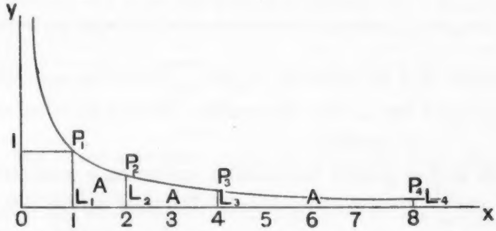

FIG. 1.

If, for example, the common ratio of the G.P. is 2, so that $OL_1=1$, $OL_2=2$, $OL_3=4$, $OL_4=8$, and so on; then if

Area $P_1P_2L_2L_1$ is A square units,

then Area $P_1P_3L_3L_1$ is $2A$ square units,

and Area $P_1P_4L_4L_1$ is $3A$ square units;

that is, each of the trapezoidal areas marked on the figure contains A square units.

The corollary of this theorem was first enunciated by Gregory of St. Vincent's follower, de Sarasa. The corollary is as follows: If the abscissae measured from the origin are in G.P., then the corresponding areas measured from the ordinate 1 (that is, from P_1L_1 in the figure) will be the logarithms of the abscissae. This theorem with its corollary is the basis of the method which I have adopted for introducing e to beginners. It was a knowledge of this theorem that led Nicolaus Mercator in 1667 to his remarkable discovery that

$$\int_1^x \frac{dx}{x} = \int_0^z \frac{dz}{1+z} = \log_e(1+z) = \int_0^z (1-z+z^2-z^3+\ldots)$$
$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots.$$

Mr. Child informs us that Barrow, Sir Isaac Newton's tutor, was conversant with Gregory of St. Vincent's theorem and made considerable use of it. But the theorem seems to have fallen into complete desuetude. There is, it is true, a reference to it in Taylor's *Ancient and Modern Geometry of Conics*, where it appears as the last of fifty or more exercises on the asymptotes of the hyperbola.

I apologise for the length of my prolegomena, but I can now describe in detail what my method is.

I commence logarithmic functions immediately after the usual elementary work on the differentiation and integration of x^n and before taking trigonometrical functions. I revise the class's knowledge of the fact that the integration of x^n according to the usual formula breaks down when $n=-1$, *i.e.* when we are trying to effect the quadrature of the equilateral hyperbola $xy=1$.

We draw the graphs of (say) $y=x^{-\frac{1}{2}}$ and $y=x^{-1\frac{1}{2}}$, *i.e.* of curves approximating in shape to the equilateral hyperbola, and apply the formula $\int x^n\,dx$.

This suggests that if we replace $\frac{1}{x}$ by $\frac{1}{x^{(1-h)}}$, where h is as small as we please, the general formula will still hold good.

Now
$$\int_1^x \frac{dx}{x^{1-h}}=\left(\frac{x^h}{h}\right)_1^x=\left(\frac{x^h-1}{h}\right).$$

If we are in too great a hurry and put $h=0$ forthwith, we get the indeterminate form $\frac{0}{0}$.

I then suggest that we might try to find $\int_1^2 \frac{dx}{x}$, *i.e.* the area $P_1P_2L_2L_1$ in Fig. 1, by making h (say) $\frac{1}{20}$ and then smaller. We find by actual calculation that $\underset{h\to 0}{\text{Lt}}\left(\frac{2^h-1}{h}\right)\to \cdot 7$ (approx.).

This result is then verified by counting squares on a good figure. The areas $\int_1^3 \frac{dx}{x}$ and $\int_1^4 \frac{dx}{x}$ are similarly found. The class notices that the $\int_1^4 \frac{dx}{x}$ gives an area 1·4 (approx.), *i.e.* that the first two trapezoidal areas in Fig. 1 are equal.

And now we are on the verge of discovering Gregory of St. Vincent's theorem: for the question naturally arises, What abscissa must we take in order to get a trapezoidal area equal to the two last? By calculation and trial it is found that the required abscissa is 8.

If we set out our results we get:

Abscissae, - - - - - - -	1,	2,	4,	8;
Area measured from ordinate 1, - -	0,	0·7,	1·4,	2·1.

Or, restating the same result in another form, we get:

Abscissae, - - - -	1,	2,	2^2,	2^3;	
Areas, - - - - -	0,	A,	$2A$,	$3A$,	where $A=0{\cdot}7$.

In other words, the abscissae form a G.P., while the areas measured from the ordinate 1 form an A.P.

Further experiments are made by varying the common ratio of the G.P. *E.g.* if the abscissae are 1, $1\frac{1}{2}$, $2\frac{1}{4}$, the areas as before will be in A.P. I now get the class to draw de Sarasa's inference that the areas can be regarded as the logarithms of the abscissae to some unknown base: for we observe that when we square the abscissae, we double the area, and when we cube the abscissae we treble the area. In other words, the areas behave like our common logarithms.

Formally our results are:

$$0{\cdot}7=\log_b 2, \quad i.e.\ 2=b^{0\cdot 7};$$
$$1{\cdot}4=\log_b 4, \quad i.e.\ 2^2=b^{1\cdot 4};$$
$$2{\cdot}1=\log_b 8, \quad .e.\ 2^3=b^{2\cdot 1}.$$

If at this stage we introduce the idea of motion, then we may say that as the moving point L generates the abscissae, the moving ordinate P_1L_1 generates areas which are the logarithms of the corresponding abscissae.

And if we imagine P_1L_1 to move parallel with itself in such a way that it takes A units of time to describe A units of area, that is, equal areas are described in equal times (time being measured from the instant when the moving ordinate leaves P_1L_1); then if

δA is an element of area,

δt is the corresponding element of time,

we have $$\delta A=y\,\delta x=\frac{1}{x}\,\delta x,$$

i.e. $$\frac{\delta x}{\delta A}=x$$

or $$\frac{\delta x}{\delta t}=x;$$

i.e. if the moving line is to describe equal areas in equal times, its velocity must increase continuously with its distance from the y-axis.

Before proceeding to find our unknown base, I draw the class's attention to the logarithmic relation existing between the terms of a G.P. whose first term is 1 and an A.P. whose first term is 0.

$$\text{G.P.}\ 1,\ r,\ r^2,\ r^3;$$
$$\text{A.P.}\ 0,\ A,\ 2A,\ 3A.$$

At this stage in the argument a slight digression into Mathematical History is not, I think, a waste of the class's time. I remind them that the first inventors of logarithms, Napier and Bürgi, did not use indices and had no conception of the base: they worked entirely with the relations between a G.P. and an A.P. Bürgi called his numbers in G.P. Black Numbers and those in A.P. Red Numbers, the common ratio of his G.P. being $(1+\frac{1}{10000})$.

His method is equivalent to letting £1 grow at Compound Interest at the rate of $\frac{1}{100}$ per cent. per annum. By taking a Compound Interest table, such as is given in any arithmetic book, it is easy to show the class that it could be used as a table of logarithms. But to make the table effective it would be necessary to take a very low rate of interest. Finally, if we made our series of numbers in G.P. more and more dense, we should gradually approach $(1+1/n)$ as our common ratio, where n approaches infinity.

May I here draw attention to a mistake which is often made in histories of Mathematics, for example, in Ball's *History* and in Glaisher's article on Logarithms in the *Encyclopaedia Britannica*? Napier's base was no more e^{-1} than Bürgi's base was e. Napier had no base. He had a growth factor $\left(1-\frac{1}{10^7}\right)$ which, if raised to the power of 10^7 will give an approximation to e^{-1}. Similarly, Bürgi's growth factor $\left(1+\frac{1}{10^4}\right)$, if raised to the power of 10^4, will give an approximation to e, and this value does, in fact, occur in Bürgi's tables. (*Vide* Tropfke, vol. ii. p. 165.)

I return from this digression to the question of the unknown base of our "natural logarithms," for that is the excellent name that Mercator gave them: the areas in the equilateral hyperbola are the logarithms, and the areas may be regarded as its "natural" progeny. Our unknown base will

also be a "natural" one, and we shall not have to fix on 10, or on $\left(1+\frac{1}{10^4}\right)$, or on any other arbitrary number as our common ratio, or growth factor or base.

We obtained previously that $2=b^{0\cdot 7}$,

i.e.
$$\log_{10}2=0\cdot 7\log_{10}b\,;$$
$$\therefore\ \log_{10}b=\frac{\log_{10}2}{0\cdot 7}=0\cdot 4343\,;$$
$$\therefore\ b=2\cdot 718.$$

I point out to the class that they ought no more to be surprised at the base of natural logarithms being 2·718... than at a radian being 57·295... degrees. Whenever in Nature we encounter periodicity, we need not be surprised if in our Mathematics we get π, and, similarly, if we are confronted by problems of natural growth we shall probably get e if we attempt to describe the growth.

e is a "natural" base because it involves a growth factor or common ratio $(1+1/n)$, where $n\to\infty$. But before proving that e is, in fact, $\underset{n\to\infty}{\mathrm{Lt}}\,(1+1/n)^n$ and that this limit can be derived immediately from Gregory of St. Vincent's theorem, I want to summarise our results.

We know now that
$$\int_1^2\frac{dx}{x}=\underset{h\to 0}{\mathrm{Lt}}\left(\frac{2^h-1}{h}\right)=\log_b 2=\log_e 2,$$
and we presume that, in general,
$$\int_1^x\frac{dx}{x}=\underset{h\to 0}{\mathrm{Lt}}\left(\frac{x^h-1}{h}\right)=\log_b x=\log_e x.$$

Before this presumption can be authenticated, we shall have to obtain a general proof of Gregory of St. Vincent's theorem.

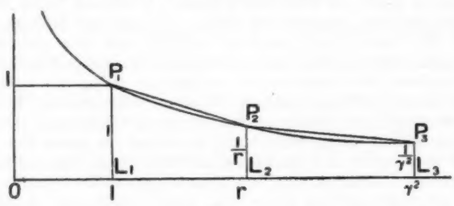

FIG. 2.

Instead of the abscissae 1, 2, 4 ..., we take the general abscissae 1, r, r^2, ...
Then the figure exhibits three different G.P.'s:

The abscissae, - - - - - - $1,\ r,\ r^2,\ r^3,\ \dots$.

The ordinates, - - - - - - $1,\ \frac{1}{r},\ \frac{1}{r^2},\ \frac{1}{r^3},\ \dots$.

Distances between successive ordinates, $(r-1)$, (r^2-r), (r^3-r^2), ...

Now area of trapezium $P_1P_2L_2L_1$ = average of parallel sides × height
$$=\frac{1}{2}\left(1+\frac{1}{r}\right)(r-1).$$

Similarly,
$$P_2P_3L_3L_2=\frac{1}{2}\left(\frac{1}{r}+\frac{1}{r^2}\right)(r^2-r).$$

These expressions are obviously equal: in general, the height of any trapezium is r times the height of the preceding trapezium, while the average of its parallel sides is $\frac{1}{r}$ times the preceding average. Hence all the trapezia are equal in area.

But not only are the trapezia equal, but so are the corresponding trapezoidal areas. To prove this:

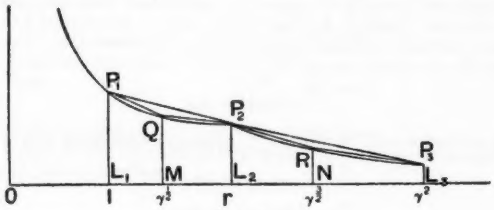

FIG. 3.

Take OM the geometric mean of 1 and r, viz. $r^{\frac{1}{2}}$, and erect the ordinate $\frac{1}{r^{\frac{1}{2}}}$. Then as $1, r^{\frac{1}{2}}, r$, form a G.P. with common ratio $r^{\frac{1}{2}}$, precisely the same argument as before proves that the trapezium $P_1QML_1 = \text{trap. } QP_2L_2M$.

Again, take ON the geometric mean of r and r^2, viz. $r^{\frac{3}{2}}$.

It is now obvious that all four trapezia in the figure are equal.

The four trapezia can be replaced by eight by making the common ratio of the abscissae $r^{\frac{1}{4}}$.

This process can be continued indefinitely, until in the limit the difference between the sum of the trapezia which can be interpolated between P_1L_1 and P_2L_2 and the corresponding trapezoidal area $\to 0$. As there will be the same number of equal trapezia between P_2L_2 and P_3L_3, and their sum will also approach the area of the corresponding trapezoidal area, it follows that these two trapezoidal areas are equal, and hence we have proved Gregory of St. Vincent's theorem.

In this proof, I am indebted for some valuable suggestions to a member of the audience before whom this paper was read.

We have proved the theorem, but we still require a deeper insight into the character of the base.

As before, let the abscissae $1, r, r^2, r^3, \ldots$, be arranged on the x-axis, and let A = area between the curve, the x-axis, P_1L_1 and PL, the ordinate at a general point $\left(x, \frac{1}{x}\right)$ on the curve. Let $x = r^n$ and $r = (1+s)$.

Now it will be noted that the figure exhibits five different groups of equal areas:

(1) The trapezoidal areas are equal (Gregory of St. Vincent's theorem).

(2) The trapezia are equal (proved previously).

(3) The segmental areas are equal (the segments are the differences between the trapezia and the trapezoidal areas).

(4) The oblongs which are severally greater than the trapezoidal areas are equal (their heights diminish in the same ratio as their bases increase).

(5) The oblongs which are severally less than the trapezoidal areas are equal.

As $x = r^n$ there are, by Gregory of St. Vincent's theorem, n equal trapezoidal areas between the first and last ordinates.

Therefore the area of any one of them, say the first, equals $\frac{A}{n}$.

Now $\dfrac{A}{n} <$ oblong P_1L_2 and $> P_2L_1$,

i.e. $\dfrac{A}{n} < 1 \,.\, (r-1)$, i.e. $< s$;

and $\dfrac{A}{n} > \dfrac{1}{r}(r-1)$, i.e. $> \dfrac{s}{s+1}$,

i.e. $s > \dfrac{A}{n} > \dfrac{s}{(s+1)}$,

i.e. $ns > A > \dfrac{ns}{(s+1)}$.

But as $s \to 0$ and $n \to \infty$, ns and $\dfrac{ns}{(s+1)}$ approach each other and A.

Now $x = r^n = (1+s)^n$;

$$\therefore\ (1+A/n)^n \to x \text{ as } n \to \infty.$$

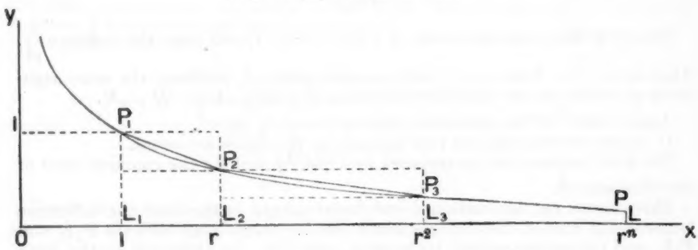

FIG. 4.

Let us interpret this result—an interpretation is necessary if this and all such results are to mean anything to the beginner. If, *e.g.*, $A = 3$ square units where the square $OP_1 = 1$ square unit, and we ask how far must the initial ordinate P_1L_1 travel parallel to itself in order that the area generated is three square units, then first of all it is obvious that the distance travelled will be finite; and we say we can find this distance by dividing 3 into n equal parts and arranging a series of points in G.P. between L_1 and L whose abscissae are: $1, (1+3/n), (1+3/n)^2, \ldots, (1+3/n)^n$.

Also the areas of the n equal oblongs severally greater than the corresponding trapezoidal areas will be $3/n$. Then, as $n \to \infty$, the sum of these n oblongs will approach 3 square units, and therefore $x \to (1+3/n)^n$.

Similarly, if $A =$ one square unit, and we ask how far must P_1L_1 travel so that the area generated may be one square unit, we get

$$x' = \operatorname*{Lt}_{n \to \infty} (1+1/n)^n.$$

But we know already that the areas are the logarithms of the corresponding abscissae, and we have found by trial that the base is $2{\cdot}718\ldots$, which we have called e.

Therefore, as 1 is the area and x' the abscissa,

$$1 = \log_e x', \quad \text{i.e. } x' = e,$$

i.e. $$e = \operatorname*{Lt}_{n \to \infty} (1+1/n)^n.$$

Again, in general, if the area is A and the abscissae x, $A = \log_e x$, *i.e.*

$$x = e^A = \text{Lt}\,(1 + A/n)^n.$$

I will conclude this paper by an alternative proof of Gregory of St. Vincent's theorem which I have given to fifth and sixth forms, but not to Polytechnic students. I do so partly to draw attention to the possibilities of the equilateral hyperbola as a field for easy exercises in Geometry and Calculus not requiring any previous knowledge of Conics, and partly to emphasise the importance of associating e with the hyperbola as firmly as π is associated with the circle. All the results obtained are simple geometrical or algebraical deductions from the fact that if (x_1, y_1), (x_2, y_2) are points on the equilateral hyperbola, then

$$x_1 y_1 = x_2 y_2 = 1.$$

I break up the proof into four separate propositions.

(1) Any secant to the equilateral hyperbola makes equal intercepts between the curve and the asymptote.

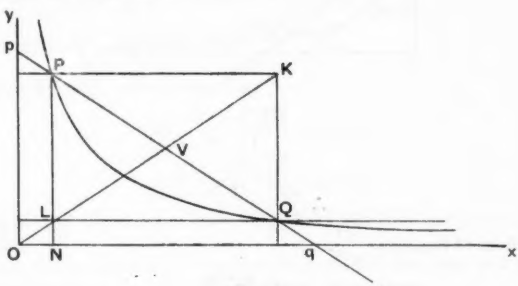

FIG. 5.

P and Q are two points (x_1, y_1) and (x_2, y_2) on the curve $xy = 1$.

$\therefore$ oblong OP = oblong OQ.

$\therefore$ O, L, K are collinear (converse of the proposition on the equality of complementary parallelograms).

It follows as an easy deduction that V is the midpoint of PQ and pq;

$$\therefore\ Pp = Qq.$$

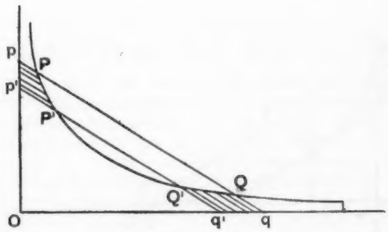

FIG. 6.

(2) If a secant move parallel to itself it will generate equal areas intercepted between curve and asymptotes.

The trapezia formed by joining P to P' and Q to Q' are equal. {By (1) and the formula for the area of a trapezium.}

And the trapezia differ from the corresponding trapezoidal areas by a quantity which $\to 0$ as the perpendicular distance between the parallel lines $\to 0$.

Hence the shaded areas can be considered as composed of an infinite number of trapezia whose perpendicular height $\to 0$.

To every elemental trapezium in the one shaded area there is a corresponding elemental trapezium in the other.

Therefore the two shaded areas are equal.

(3) If the tangent at a point (x_3, y_3) is parallel to the chord joining (x_1, y_1) and (x_2, y_2), then x_3 is the geometric mean of x_1 and x_2.

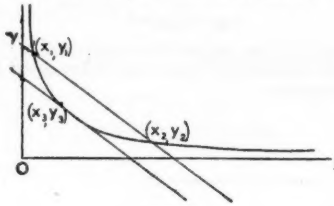

FIG. 7.

$$y = \frac{1}{x};$$

$$\therefore \frac{dy}{dx} = \frac{-1}{x^2};$$

$\therefore$ Gradient at (x_3, y_3) is $\dfrac{-1}{x_3^2}$.

Now $$\text{gradient of chord} = \frac{y_1 - y_2}{x_1 - x_2},$$

i.e. $$\text{gradient of chord} = \frac{\frac{1}{x_1} - \frac{1}{x_2}}{x_1 - x_2} = -\frac{1}{x_1 x_2};$$

$$\therefore x_3^2 = x_1 x_2.$$

(4) If the abscissae of three points on the curve are in G.P., then the two

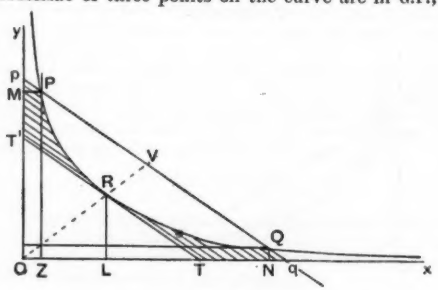

FIG. 8.

areas enclosed by the curve, the x-axis and the corresponding three ordinates are equal (this is Gregory of St. Vincent's theorem).

Now triangle PpM = triangle QqN;

$\therefore$ $MPRT' = NQRT$ (subtracting the triangles from the areas proved equal in (2).

Now triangle ORT = triangle ORT';

$$\therefore ORQN = ORPM,$$

i.e. $$ORL + RQNL = MPRLO - ORL,$$

i.e. $$RQNL = MPRLO - 2ORL.$$

But $$2ORL = x_3y_3 = x_1y_1 = MPZO;$$

$$\therefore RQNL = PRLZ,$$

i.e. areas enclosed by curve, asymptote and successive ordinates are equal.

I conclude by recapitulating the steps in this method, which are fewer and simpler than would at first sight appear from the length of this exposition—intended, of course, for the teacher and not for the pupil.

(1) Obtain the area $\int_1^x \frac{dx}{x}$ by integrating $\int_1^x \frac{dx}{x^{(1-h)}}$.

This integral $= \underset{h \to 0}{\mathrm{Lt}} \left(\frac{x^h - 1}{h}\right)$.

(2) Show by actual calculation that

$$\underset{h \to 0}{\mathrm{Lt}} \left(\frac{2^h - 1}{h}\right) = 0{\cdot}7 \text{ (approx.)};$$

then $$\underset{h \to 0}{\mathrm{Lt}} \left(\frac{4^h - 1}{h}\right) = 1{\cdot}4 \text{ (approx.)}.$$

(3) By similar experiments show that if the abscissae are in G.P. the areas are in A.P.

(4) Deduce the logarithmic relation between the abscissae and the areas, and state Gregory of St. Vincent's theorem.

(5) Calculate the unknown base, and show that it is 2·718....

(6) Prove Gregory of St. Vincent's theorem by replacing the equal trapezia by equal trapezoidal areas.

(7) Deduce from this proof that

$$e = \underset{n \to 0}{\mathrm{Lt}} \left(1 + \frac{1}{n}\right)^n$$

and $$e^A = \underset{n \to 0}{\mathrm{Lt}} \left(1 + \frac{A}{n}\right)^n.$$

J. KATZ.

GLEANINGS FAR AND NEAR.

331. (Buffon), so some of his biographers assert, became so wedded to his geometry that while his companions were at their sports he was generally to be seen in some retired nook poring over his pocket *Euclid*, which he seems to have cherished at this early age with no less affection than Parson Adams had for his *Æschylus*.... There are stories current that he had anticipated Newton in some of his discoveries, but that he withheld his claim, observing that people were not obliged to believe the assertion. We receive these *on dits* with some grains of allowance, for to say nothing of dates, vanity was certainly not absent as an ingredient in Buffon's character.—"Buffon," *Penny Cyclopaedia*.

332. The crest line runs East and West... being bisected into two equal parts by the road.—Sir Evelyn Wood.

GEOMETRIC SOLUTION OF THE QUADRATIC EQUATION.

By Prof. G. A. Miller, Ph.D.

The main object of the present note is to direct attention to the fact that Lill's method for solving geometrically the general quadratic equation with real coefficients can be used effectively for the purpose of impressing upon students that even in elementary mathematics some very useful advances have been made recently. In 1867 E. Lill published an article in volume 65 of the Paris *Comptes Rendus*, from which the following method for solving the equation

$$x^2 - ax + b = 0$$

can easily be deduced. Construct the circle whose diameter is the line segment which joins the points $(0, 1)$ and (a, b). When this circle cuts the x-axis in real points the abscissae of these points are the roots of the given quadratic equation. When it touches the x-axis the abscissa of the point of contact represents the value of its equal roots. In all other cases the roots are imaginary, and are represented by the numbers of the complex plane which correspond to the points of intersection of the line $x = a/2$ and the circle whose centre is at the origin and whose radius is equal to the length of the tangent line from the origin to its point of contact with the given circle. The proof by elementary geometry is obvious.

The modernness of a part of our elementary algebra may be further emphasized by a note due to the late G. Eneström, which begins substantially as follows: "Properly speaking one does not meet among the Greeks the least trace of a really graphic calculus. It is often said that they solved the equation of the second degree geometrically, but in reality these solutions concerned always certain problems of geometry which are not truly graphic solutions of equations of the second degree, even if they can be reduced to the solution of equations of the second degree when one makes use of our modern algebraic notation." *Encyclopédie des Sciences Mathématiques*, tome 1, volume 4, p. 340.

Emphasis should be laid on the fact that the method due to Lill gives also a simple geometric construction of the roots when they are imaginary, while the ancient Greeks knew nothing about imaginary roots, and never gave more than one root. They solved no quadratic equation having two negative roots, and hence it seems that the common statement to which Eneström referred, and which appears in some of our most recent histories of mathematics, viz. that the Greeks solved the quadratic equation geometrically, is apt to convey an incorrect notion to the student unless this statement is properly modified. It would be much more accurate and more impressive to say that the ancient Greeks began the geometric solution of the quadratic equation, but this solution was not completed before the nineteenth century. While accurate statements relating to large questions in mathematical history present a difficult goal, the effort to formulate such statements has great educative value for the student.

In view of the well-known fact that the Greeks considered irrational quantities long before they considered irrational numbers, the student might be led to wonder whether the statement that they solved the quadratic equation geometrically might not imply that they also considered negative and imaginary quantities long before they considered negative and imaginary numbers. One of the best illustrations of a negative quantity is a debt, and it is likely that in this sense the ancients knew something of negative quantities before they began to use negative numbers, but we have no evidence of any geometric constructions along this line on the part of the ancient Greeks. With respect to imaginary numbers, there is good reason to assume that the number concept preceded the quantity concept, even if

number pairs were considered long before complex numbers were used. Such considerations have, however, no direct bearing on the statement that the Greeks solved the quadratic equation geometrically.

The so-called geometric solution of the quadratic equation by the Greeks may be illustrated, in part, by the study of the Golden Section in Euclid's *Elements*. A given straight line is to be cut so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment. This is obviously equivalent to a solution of the quadratic equation $a(a-x)=x^2$. As in other cases, Euclid gives only one solution. In fact, the second solution of the present case is clearly negative, but this is not always true when Euclid and other Greek writers gave only one solution. There is obviously a wide difference between the considerations of this geometric problem, whose solution is needed in the construction of a regular pentagon, and the consideration of the solution of the general quadratic equation with which it is now naturally associated.

G. A. MILLER.

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333. Perveniri ad Summum nisi ex Principiis non potest.

NEWTONUM ingentem, lumen non unius aevi,
 A, B quae docuit prima, magistra fuit.
Doctior Ille statim vetulâ, cito sensit inani
 Quiddam his literulis majus inesse sono.
Protinus egregios elementis repperit usus;
 Usus, quos nunquam conjiciebat anus.
Notosque ignotis numeros conferre peritus,
 Inde potestates format utrisque datas.
Laudo tamen vetulae praecepta ea primula, quaeque
 Newtoni haud dubitem dicere Principia.—Vincent Bourne.

Great Newton's self, to whom the world's in debt,
Owed to Schoolmistress sage his Alphabet;
But quickly wiser than his teacher grown,
Discovered properties to her unknown;
Of A *plus* B, or *minus*, learned the use,
Known Quantities from unknown to deduce;
And make—no doubt to that old dame's surprise—
The Christ-Cross-Row his ladder to the skies.
Yet whatsoe'er Geometricians say,
Her lessons were his true PRINCIPIA!

334. Among the volumes in the De Morgan collection in the London University Library is one lettered on the back:—Collins. *Commercium Epistolicum.* 1712. This contains many other items, the first of which is a printed form from the Royal Society, stating that Mr. De Morgan's "Account of some recent Discoveries in England and Germany relative to the controversy in the Invention of Fluxions" has been duly received, and that the Society "duly appreciate this mark of consideration" [underlined thus by De M.]. Beneath this, De Morgan has written:

"Not yet—by and bye perhaps. A. De M."

This form is dated Dec. 19. 1851, and the article sent appeared in *The Companion to the Almanac*, Part I., 1852.

Another form of receipt, dated Nov. 18, 1848, for an article from the *Phil. Mag.*—"On the Additions made to the *Commercium Epistolicum*"—has a pencilled comment on the "due appreciate":

"They ought to do so if they have any gratitude, for I do not give them a quarter of their deserts. A. De M."

MATHEMATICAL NOTES.

805. [V. 1. a. μ.] *A Curiosity in Elementary Mechanics.*

If the value of $g = 31{\cdot}998$ ft./sec.^{2},

then $\quad$ 1 joule $= 0{\cdot}74162$ ft.-lb.,

and $\quad$ 1 HP $= 0{\cdot}74162$ kwt.

The exact statement of this is as follows. If 1 cm. $= L$ ft., and 1 gm. $= M$ lb., then

$$\frac{1 \text{ joule}}{1 \text{ ft. lb.}} = \frac{1 \text{ HP}}{1 \text{ kwt.}} = \sqrt{0{\cdot}55},$$

provided
$$g = \frac{L^2 M \times 10^8}{\sqrt{55}} \text{ ft./sec.}^2$$

In the above calculation we have taken

$$L = 3{\cdot}280843 \times 10^{-2} \quad \text{and} \quad M = 2{\cdot}204622 \times 10^{-3}.$$

Also 1 HP $= 550$ ft.-lb./sec.

D. M. Y. SOMMERVILLE.

806. [T. 3. a.] *Deviation of a Refracted Ray of Light.*

[While it is probable that few if any of the constructions here described can be new, it is believed that they are not commonly shown in text-books in their connexion as here. They are the outcome of the solution of a problem on tangents to coaxal circles, which did not at first sight seem to have any bearing on Geometrical Optics.]

Let the plane of the paper be that of incidence; $X'OX$ its intersection with the refracting surface; OY the normal at the point O of incidence. Along OX take $OA : OB$ equal to $1 : \mu^2$, and let the incident ray cut the circles on OA, OB as diameters in P, Q, and let the ordinate MP cut circle OQB in R.

(i) The refracted ray passes through R.

For
$$OR^2 = OM \, . \, OB\,; \quad OP^2 = OM \, . \, OA\,;$$
$$\therefore \; OR^2 = \mu^2 OP^2 \quad \text{and} \quad OR = \mu OP\,;$$
$$\therefore \; \frac{\sin YOP}{\sin YOR} = \frac{\sin OPM}{\sin ORM} = \frac{OR}{OP} = \mu.$$

(ii) The deviation increases with the incidence.

Produce AP, BR to meet in D, and join OD.

$\because \; O\hat{P}D$, $O\hat{R}D$ are right, $\;\therefore\; O, P, R, D$ are concyclic.

$\therefore \; O\hat{D}A = O\hat{R}M = O\hat{B}D$, $\;\therefore\; OD$ touches circumcircle of $\triangle\, ABD$.

$\therefore \; OD^2 = OA \, . \, OB = \mu^2 OA^2$.

$\therefore \; OD = \mu OA$ and locus of D is a circle.

Let OB, BD cut this circle in E, F.

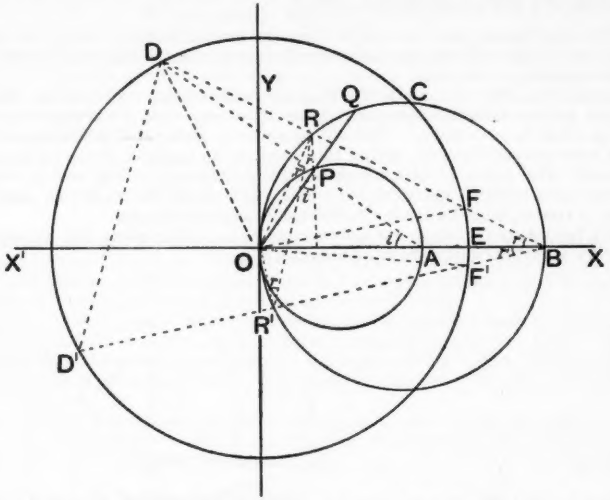
Then $O\hat{F}D = O\hat{D}F = O\hat{P}M$, since O, P, R, D are concyclic and $O\hat{B}D = O\hat{R}M$.

$\therefore \; B\hat{O}F = P\hat{O}R =$ deviation, which therefore is measured by the arc EF and increases with the angle of incidence.

(iii) To find the total deviation on passing through a prism.

Let the first angle of incidence be YOQ as shown. Make $OBD' =$ angle of incidence at the second face. Let BD' cut circle DFE in F', D' as shown. Then $OF'D'$ is the angle of emergence; $\therefore\; FOF'$ is the total deviation.

(iv) If in the above figure we take DBD' as the normal section of the prism, RR' will represent the path within the prism of a refracted ray arising from one



incident *at R' with angle* $i = YOP$, the emergent ray at R making $i' = OBD$ with the normal OR.

For $$O\hat{R'}R = O\hat{B}R = r,$$
$$O\hat{R}R' = O\hat{B}R' = r',$$

as in the corresponding part of the figure for a somewhat different treatment by Mr. F. G. Smith in his article on "Refraction through a Prism," in *The Optician* for April 1918. It is to be noted that when use is made of DBD' *as a normal section of the prism*, the ray incident at R' would pass through Q, and that the ray emergent at R could be similarly found.

A construction for finding the angle of incidence when μ and the refracting angle of the prism are known may be easily deduced. E. M. LANGLEY.

807. [V. 1. a. μ.] "*Direction*" *and* "*Dimensions.*"

May I draw attention to the value of considering "direction" as well as "dimensions" when analyzing an expression dealing with vectors?

To take a simple case, the range of a projectile on a horizontal plane, viz.
$$2V^2 \sin a \cos a \div g.$$

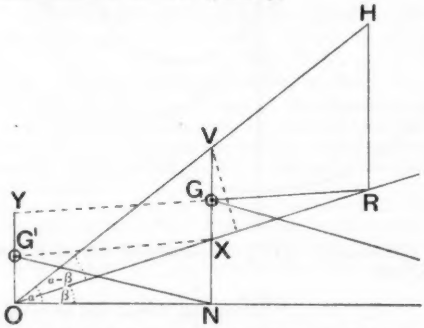
Now this denotes a horizontal vector, and yet its denominator contains g, which is a vertical vector. Therefore the numerator must consist of the product of two vectors, the one vertical to balance against g, and the other horizontal. We thus separate the expression into $V \cos a \times \dfrac{2V \sin a}{g}$, where $V \cos a$ is horizontal, viz. the horizontal distance travelled per second, and $\dfrac{2V \sin a}{g}$ is a non-vector, of the dimension of time, giving the time required for the body to reach the ground again.

If, further, we look at the *dimensions* of each part, we can get at the true reason for the formula. Turn each factor into "length" by judicious introduction of t, and we get $Vt\cos\alpha \times \dfrac{Vt\sin\alpha}{\frac{1}{2}gt^2}$.

The first factor gives the range when the second factor is unity, *i.e.* when the rise $Vt\sin\alpha$ due to the initial conditions is exactly balanced by the fall due to gravity.

Note that, since t cancels, the formula holds for all values of t. This is worth further consideration, as it leads to a very neat *construction* for the range when V, α are given. This will, however, be elaborated in connexion with the more general formula, giving the range on an inclined plane, of slope β, through the point O of projection, $2V^2g^{-1}\cos\alpha\sin(\alpha-\beta)\sec^2\beta$, which seems much more complicated, but which yields its secrets readily on analysis, and, of course, includes the horizontal range as a special case.

We have here to search for a factor giving a vector along the plane, and also for a vertical vector to balance against g.



Now $V\cos\alpha$ is horizontal, viz. ON in the diagram, if OV represents V, and $V\cos\alpha\sec\beta$ is OX, which is along the plane, and is the distance travelled along the plane in one second by a point which keeps vertically in line with the actual projectile, its shadow, so to speak, if the sun were in the zenith.

The factor $V\sin(\alpha-\beta)$ is represented by the dotted perpendicular from V on to the plane, and $V\sin(\alpha-\beta)\sec\beta$ gives the vertical line VX, and this $\div\frac{1}{2}g$ is of the dimensions of time, being, in fact, the time needed to reach the plane again. Note that $\frac{1}{2}g$ is the distance fallen under gravity in the first second; therefore if VG in the diagram represents $\frac{1}{2}g$, G is the position of the projectile at that moment.

The formula is therefore $\dfrac{OX \,.\, VX}{VG}=$ the range OR.

It is easy now to construct for OR.

Draw OG' vertically from O, equal to GV, and join $G'X$. Then the line through G parallel to $G'X$ will pass through R.

For, producing RG to cut OG' in Y, we have $OY=VX$, and $\dfrac{OR}{OY}=\dfrac{OX}{OG'}$, which verifies the construction.

As a particular case, the horizontal range is obtained by drawing through G a line parallel to $G'N$.

Note that the time required is obviously $\dfrac{OR}{OX}$, since OX was described in 1 second, and $\dfrac{VX}{VG}=\dfrac{OR}{OX}$, so that the formula $OX \,.\, \dfrac{VX}{VG}$ is justified.

If, further, we introduce any time t into the formula, as in the previous case, it becomes

$$Vt\cos\alpha\sec\beta\times\frac{Vt\sin(\alpha-\beta)\sec\beta}{\frac{1}{2}gt^2},$$

where the fraction has a numerator showing how high above the plane the projectile would be if there were no gravity, and a denominator which shows how far gravity has dragged it down below that height. When these are equal, the first factor, $V.t\cos\alpha\sec\beta$, equals the range, obviously. Also, from the Fig., with this value of t,

$$HR=\frac{OR}{OX}.VX=t.VX, \text{ also } VX=t.VG;$$

$$\therefore\ HR=\tfrac{1}{2}gt^2.$$

But the formula holds whatever t may be. This means that, in the diagram we may make $OV=V.t$, $VG=G'O=\frac{1}{2}gt^2$, then $VX=t.V\sin(\alpha-\beta)\sec\beta$, and the construction is unimpaired. All that is needed is that G shall be some position of the projectile and G' placed so that $OG'=GV$.

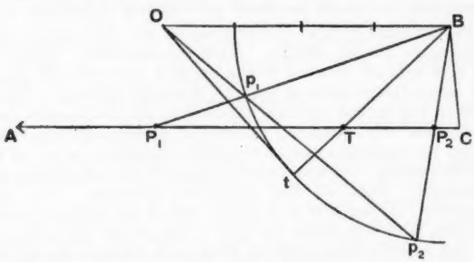
Any number of points on the parabola can be quickly found by drawing various lines OX, and drawing lines from G parallel to the $G'X$ lines. The corresponding times are proportional to the values of VX. But, perhaps, the quickest way of all is to ignore G', and merely make $OY=VX$ and draw YGR to cut OX in R.

The obvious homography of the pencils from O and G is worth noting; so also is the fact that the X and Y points form two equal parallel time scales, one down from V, the other up from O.

A. LODGE.

808. [X. 4.] *A Problem with Solution.*

A man wishes to go from a point A on a straight road to a point B in a field on one side of the road. He walks along the road at 4 miles per hour, and in the field at 3 miles per hour. He walks to a point P on the road, and then in a straight line from P to B in the field. Show how to construct geometrically pairs of such paths which will take equal times, and, incidentally, the path of shortest time. The distance from A to C, the point on the road nearest to B, may be taken as, say, 200 yards, and $BC=50$ yards, to ensure that AC is long enough to prevent the path of shortest time from being straight from A to B. There is no need for A to appear in the diagram.



The construction for the solution is as follows:

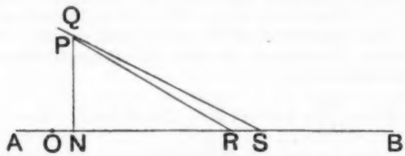
On any scale draw a circle of 3 units radius, centre B, and take a point O, 4 units from B, so that BO is $\parallel$ to CA and codirected.

Any chord Op_1p_2 is such that Bp_1, Bp_2 cut AC in points P_1, P_2, which give paths of equal time; and therefore the tangent, Ot, gives rise to Bt, cutting AC at T, where he must turn off for the shortest time.

A. LODGE.

809. [K^1. 10. e.] *Problem, and Solution.*

A thread is lightly gummed along a fixed straight rod, and is then slowly pulled off the rod by one end, the pull at any instant being in the direction of the portion already pulled off; to find the locus of the end of the string.



Let AB be the fixed rod, PR the portion of string already pulled off, RB the portion still gummed to the rod. When P has been pulled a very short distance to Q in the direction RP, the string will be in the position QSB. Take the origin at O on AB, where $OR=PR$, and let P be (x, y) and Q $(x+\delta x, y+\delta y)$; also let the angle ORP be θ.

Then, since RPQ is straight, $-\dfrac{dy}{dx}=\tan\theta$, and triangle ORP is isosceles;

$$\therefore\ x=y\tan\frac{\theta}{2}.$$

Hence $-\dfrac{dy}{dx}=2\dfrac{x}{y}\Big/\left(1-\dfrac{x^2}{y^2}\right)$, or putting $x=vy$, and reducing, $\dfrac{dy}{y}=-2v.dv/1+v^2$;

whence $\log y(1+v^2)=$ a const., or say $x^2+(y-c)^2=c^2$; the equation to a circle touching OR at O and PR at P. It is now obvious geometrically that OS and QS are the tangents to this circle from S, OQ being their chord of contact.

G. Osborn.

810. [B. 1. b.] *Note on Sign of a Partial Product in Expansion of a Determinant.*

In teaching determinants, I have found the following rule for finding the sign of a partial product much simpler and easier of application than the rule given in the ordinary text-books. The proofs of the elementary propositions about determinants can easily be based on this new definition.

Rule. The elements of the partial product are, of course, such that no two are in the same line or column. Mark the elements. From the marked letter in the first column proceed to the marked letters in all the succeeding columns and count the number of ascents, ignoring descents. Then, from the marked letter in the second column, proceed to all the marked letters in the succeeding columns only, counting number of ascents and ignoring descents. Similarly, from the marked letter in the third column, proceed to the marked letters in the succeeding columns, counting the number of ascents and ignoring descents. Similarly treat all the remaining columns. Now add up the number of ascents. If the sum is odd, the sign of the partial product is negative. If the sum is zero or even, the sign of the partial product is positive.

To prove the interchangeability of rows and columns in all propositions regarding determinants, another definition is given for the sign of a partial product, and the two definitions are easily proved to be equivalent. Take the marked letters according to rows instead of columns, and instead of ascents count the number of passages from right to left, ignoring passages from left to right.

Vizianagram, India. S. Purushtham

811. [K^1. 11. e.] *A Generalisation of Feuerbach's Theorem.*

The following theorem includes that of Feuerbach, as well as several known theorems on pedal circles: Let $A_1A_2A_3$ denote any euclidean plane triangle, O its circumcentre, O_i the midpoint of A_jA_k, P, Q any two points of the plane, P_i, Q_i the orthogonal projections of P, Q respectively on A_jA_k; let $(P, Q)_i$ denote the circle, centre O_i, with respect to which the points P_i, Q_i are inverse, and let (P, Q) denote the circle which is orthogonal to the three circles $(P, Q)_i$ $(i, j, k=1, 2, 3)$. Then the circle (P, Q) meets the nine-point circle of $A_1A_2A_3$ at the orthopoles of the lines OP, OQ.

(i) First, if S denote a point on the circumcircle of $A_1A_2A_3$, then the orthopole of OS is on the circle $(S, S)_i$. For if S' denote the point diametrically opposite to S on the circumcircle, then $(S, S)_i$ is the circle on the segment S_iS_i' as diameter; also the pedal lines, with respect to $A_1A_2A_3$, of S, S' meet at right angles on the nine-point circle of the triangle, at the orthopole of OS; etc.

(ii) Again, if T denote any point on the circumcircle, and D a point, on that circle, such that the diameter OD is perpendicular to any side of the triangle $A_1A_2A_3$; then the three circles $(S, T)_i$ are all real if and only if none of the six points D is on the minor arc ST of the circumcircle. This case only being here discussed, it will be shewn that the orthopoles, M, N, of OS, OT respectively are on the circle (S, T). For there is a circle, c, with centre at the midpoint of the minor arc MN of the nine-point circle, and passing through the points M, N. Let the line O_iM cut the circle c again at N'; then $O_iN \equiv O_iN'$, and thus the square of the length of the tangent from O_i to c is measured by $O_iM \,.\, O_iN$; also O_iM, O_iN are respectively congruent to O_iS_i, O_iT_i. The circle c therefore cuts orthogonally the three circles $(S, T)_i$, and is thus the circle (S, T).

(iii) Finally, if P, Q denote variable points on the fixed lines OS, OT respectively, then the ratios

$$O_1P_1 \,.\, O_1Q_1 : O_2P_2 \,.\, O_2Q_2 : O_3P_3 \,.\, O_3Q_3$$

are fixed; thus the ratios of the lengths of the tangents from O_1, O_2, O_3 to the circle (P, Q) are fixed, and the circles (P, Q) are therefore coaxial. This completes the proof of the theorem.

(iv) It should be noted that if P, Q respectively denote variable points on the fixed *half*-lines OS, OT, proceeding from O, then the circle (P, Q) is determined by the product $OP \,.\, OQ$; and in particular that, for all positions of P, (O, P) is the nine-point circle of $A_1A_2A_3$.

(v) If O, P, Q be in line, then the circle (P, Q) touches the nine-point circle at the orthopole of OP; in particular, the circle (P, P) touches the nine-point circle at that point. If P denote successively the in- and excentres of $A_1A_2A_3$, then Feuerbach's theorem is obtained.

(vi) If P, Q be such that the six points P_i, Q_i are on the same circle, then that circle is (P, Q). The main theorem, and the special case (v), thus yield known theorems on pedal circles.

(vii) If the restriction that O denotes the circumcentre be removed, while O_i denotes the orthogonal projection of O on A_jA_k $(i, j, k=1, 2, 3)$, then it may be proved as in (iii) that, if P, Q denote variable points respectively on two fixed lines through O, then the circles (P, Q) cut the pedal circle (with respect to $A_1A_2A_3$) of O at two fixed points, M, N. But rotation about O of *either* of the lines OP, OQ, while the other remains fixed, involves, in general, variation of *both* the points M, N. Indeed, it is well known that, if P denote a variable point on any fixed line, p, then the pedal circle of P is orthogonal to a fixed circle, the J-circle of p. It follows that the points M, N are the limiting points of the coaxial system determined by the J-circles of OP, OQ; and thence that the circle (P, P) does not in general touch the pedal circle of O, unless O is the circumcentre.

J. P. Gabbatt.

812. [X. 1.] *Very Simple Mean-ordinate Rules.*

Let the function be $y=a_0+a_1x+a_2x^2+\dots$.

Then the mean ordinate for the interval 0 to 1 is

$$a_0+\tfrac{1}{2}a_1+\tfrac{1}{3}a_2+\dots.$$

(*a*) To find a 2-ord. rule.

We have
$$\frac{1}{2}\left[\frac{a}{n}+\left(1-\frac{a}{n}\right)\right]\equiv\frac{1}{2},$$

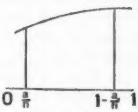


FIG. 1.

and we must have at least

$$\frac{1}{2}\left[\left(\frac{a}{n}\right)^2+\left(1-\frac{a}{n}\right)^2\right]=\frac{1}{3}$$

and
$$\frac{1}{2}\left[\left(\frac{a}{n}\right)^3+\left(1-\frac{a}{n}\right)^3\right]=\frac{1}{4}.$$

These two equations are equivalent, giving $n/a=3+\sqrt{3}=5$ (say).
Two applications of this give Dufton's rule.

(*b*) To find a 3-ord. rule.

Here
$$\frac{1}{3}\cdot\frac{(n-a)+n+(n+a)}{2n}\equiv\frac{1}{2},$$

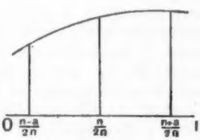


FIG. 2.

and we must have at least

$$\frac{1}{3}\cdot\frac{(n-a)^2+n^2+(n+a)^2}{4n^2}=\frac{1}{3},$$

and
$$\frac{1}{3}\cdot\frac{(n-a)^3+n^3+(n+a)^3}{8n^3}=\frac{1}{4}.$$

These two equations are again equivalent, both reducing to $n/a=\sqrt{2}$, and the best approximation for $\sqrt{2}$ is 7/5.

If $p_n=\dfrac{1}{3}\cdot\dfrac{2^n+7^n+12^n}{14^n}$, we have

$$p_1=\tfrac{1}{2},\quad p_2=\tfrac{1}{3}(1+\tfrac{1}{196}),\quad p_3=\tfrac{1}{4}(1+\tfrac{1}{98}),\quad p_4=\tfrac{1}{5}(1+\tfrac{1}{222}),\quad p_5=\tfrac{1}{6}(1-\tfrac{1}{81}),\ \text{etc.}$$

Compare with Dufton's rule, taking

$$q_n=\frac{1}{4}\cdot\frac{1^n+4^n+6^n+9^n}{10^n}.$$

Here

$$q_1=\tfrac{1}{2},\quad q_2=\tfrac{1}{3}(1+\tfrac{1}{200}),\quad q_3=\tfrac{1}{4}(1+\tfrac{1}{100}),\quad q_4=\tfrac{1}{5}(1+\tfrac{1}{70}),\quad q_5=\tfrac{1}{6}(1+\tfrac{1}{56}),\ \text{etc.}$$

There is therefore not much to choose between them: indeed the 3-ordinate rule is often more accurate than the 4-ordinate, as the following table illustrates:

Integral.	True value.	By Dufton's rule	By 3-ord. rule.
$\int_0^1\frac{dx}{1+x}$.	·69315.	·69367.	·69338.
$\int_0^1\frac{dx}{1+x^2}$.	·7854.	·7850.	·7855.
$\int_0^1 10^x dx$.	3·9086.	3·9238.	3·9163.
$\int_0^1\log_{10}(1+x)dx$.	·16777.	·16760.	·16764.

N. M. GIBBINS.

813. [L^1. 1. a.] *On Note* 796, *Gazette*, xii. p. 470.

Professor Hill's note headed "The Reduction of the Equation of a Central Conic to its Simplest Form," suggests that if m be the gradient of *either* principal axis of the conic $ax^2+2hxy+by^2=1$, and if the lines $y=mx$ and $y=-x/m$ be chosen as X and Y axes respectively, then the equation of the conic becomes

$$(a+mh)X^2+(b-mh)Y^2=1,$$

where m is *either* root of the equation $\dfrac{2m}{1-m^2}=\dfrac{2h}{a-b}$.

That $A=a+mh$, using Professor Hill's notation, can be proved directly as follows:

$$\begin{aligned} A &= a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta \\ &= \frac{a+2mh+m^2b}{1+m^2} \\ &= a+\frac{2mh-m^2(a-b)}{1+m^2} \\ &= a+mh, \quad \text{since } m^2(a-b)=m(1-m^2)h. \end{aligned}$$

If this method be adopted, *the obvious thing is done*, without, I think, sacrificing any advantages. Observe that the quadratic for m is equivalent to

$$(a+mh)(b-mh)=ab-h^2.$$

W. J. DOBBS.

12 Colinette Road, Putney, S.W. 15.

814. [L^1. 1. a.] *On Note* 796, *Gazette*, xii. p. 470.

While admitting the elegance of Prof. Hill's method, I do not see the necessity for it. The ordinary method, given in such texts as C. Smith's *Conics*, gives the position and lengths of the axes of the central conic, *and the position of the conic with regard to them*, without any ambiguity.

Step 1. (not referred to in the note). Find the centre, and reduce

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

to

$$ax^2+2hxy+by^2+\Delta/(ab-h^2)=0,$$

and, by dividing across, to the form

$$Ax^2+2Hxy+By^2=1.$$

This assumes that $ab-h^2$ is not zero. (See below.)

Step 2. The circle $x^2+y^2=r^2$ cuts this conic, referred to coordinate axes through its centre, in four points which lie on two straight lines through the origin, given by

$$(A-1/r^2)x^2+2Hxy+(B-1/r^2)y^2=0.$$

If r is the length of either semi-axis, these lines coincide, and the expression on the left-hand side becomes a perfect square. The *square-root of the expression put equal to zero* is the *equation of the axis* corresponding to the particular value of r that is taken.

Thus, if r_1^2, r_2^2 are the values of r^2 satisfying

$$(A-1/r^2)(B-1/r^2)=H^2,$$

the axis whose length is $2r_1$ *lies along the line*

$$(A-1/r_1^2)x+Hy=0.$$

If the two values of r^2 are positive, the curve is an ellipse; for its construction it is useful to sketch in the perpendiculars to the axes at their ends, thus forming a circumscribing rectangle.

If one of the values of r^2, say r_1^2, is *negative*, then the curve is a hyperbola, and the *imaginary* or conjugate axis is $(A-1/r_1^2)x+Hy=0$; and, as the asymptotes are the diagonals of the circumscribing rectangle, the curve can be sketched in at once. There is no ambiguity.

Most, or all, of the above is in the ordinary texts; the following treatment of the parabola I do not remember having seen; and it may therefore be of interest.

If $ab-h^2$ is zero, the curve is a parabola, and the equation reduces to the form

$$(lx+my)^2+ux+vy+w=0\,;$$

and the tangent at the vertex and the axis can be found in the ordinary text way.

Now from the form of the equation, all values of x, y, which represent points on the curve, must make $ux+vy+w$ *negative*; hence the curve lies wholly on one side of the line $ux+vy+w=0$; and the origin lies on the same side or on the opposite side according to the sign of w. This is sufficient to settle immediately the lie of the curve.

It should be noted in the above remarks for the central conic that the case when both values of r^2 are negative does not arise; for this would demand that $A+B$ and H^2-AB should both be negative.

Since
$$\begin{aligned}(Ax+Hy)^2+(Hx+By)^2+(AB-H^2)(x^2+y^2)\\ =(A+B)(Ax^2+2Hxy+By^2)\\ =A+B,\end{aligned}$$

this demand cannot be satisfied. J. M. CHILD.

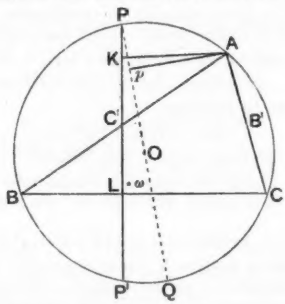
815. [K[1]. 2. a.] *Feuerbach's Theorem.*

Ap is the perpendicular from A on any diameter POQ of circumcircle.

ω is the image of p in $B'C'$, the mid-join of AB, AC.

To prove that the pedal lines of P and Q intersect at ω.

Draw chord PLP' perp. to BC and AK perp. to PP'.



Then, by symmetry, $\angle PL\omega=\angle LKp$
$$\begin{aligned}&=\angle PAp \quad (P, K, p, A \text{ concyclic})\\ &=\angle PQA \quad (\angle PAQ=90^\circ)\\ &=\angle PP'A.\end{aligned}$$

$\therefore$ $L\omega$ is par[l] to $P'A$ and is $\therefore$ the pedal line of P.

Similarly ω lies on the pedal line of Q; $\therefore$ the pedal lines of P and Q intersect at ω, a point on the 9-point $\odot$.

Now if POQ is chosen as the diameter through I, the incentre, then ω is the Feuerbach point. [See *Mathematical Gazette*, Dec. 1923, p. 421.]

Hence the pedal lines of the extremities of the diameter OI intersect at the Feuerbach point.

E. P. LEWIS.

REVIEWS.

Calculus of Variations. By G. A. BLISS. Carus Mathematical Monographs. Pp. 189. $2. 1925. (Open Court.)

The Calculus of Variations has a bad reputation in England among both students and lecturers.

In my undergraduate days at Cambridge, when little attention was paid to rigorous proofs in analysis, and we had the habit of "proving" results which were "generally" true, and of afterwards considering rather at haphazard exceptional cases in which the results obtained were absurd, it was possible without undue labour to obtain a differential equation corresponding to a reasonably simple problem of variations, and to verify that the number of constants, which the integral of the differential equation might be expected to contain, was equal to the number of conditions imposed by the boundary conditions. In some cases it was possible to integrate the equation in terms of elementary functions and even to determine the constants explicitly. We were generally content to assume as "obvious" that the magnitude with which we were concerned necessarily had a minimum (or maximum) given by the solution obtained; and if the distinction between the two cases was not "obvious," some of us could use Legendre's criterion to distinguish between them. We were perhaps a little perplexed to find that in certain cases (*e.g.* that of an arc of a great circle of length greater than π) the "solution" obtained by the usual process did not solve the problem required. On the whole I do not think that the subject was regarded as specially unattractive, and it gave solutions of a number of interesting geometrical and mechanical problems.

This comparatively comfortable state of affairs has been completely changed by a more modern outlook on pure mathematics, and especially on analysis. Any competent mathematical student of the present day is familiar with the idea that, if a proposition is "proved," there should be no further question as to whether it can ever be untrue. He realises also that if a function is to be subjected to the ordinary processes of analysis it must be in some way restricted, so as to have, for example, such properties as continuity and differentiability; and he is acquainted with quite simple variables and functions which have no greatest or least values.

To any one, whether student or lecturer, with a mathematical conscience, as I may call it, thus developed, even the simplest problems of the Calculus of Variations present a new and more formidable appearance.

In the simplest type of problem, expressed analytically, we are concerned with an integral $I \equiv \int F(x, y, p)dx$, p being the derivative of y with respect to x, and F a given function of the three variables x, y, p, while at the upper and lower limits of the integral x and y are given or in some other way restricted by so-called boundary conditions. Our problem, stated loosely, is to determine y as a function of x, $y = f(x)$, so that for this choice of y, I is, say, less than for any other choice of y, the boundary conditions being satisfied.

In order that the problem may have any meaning the integral I must exist; this requirement imposes conditions such as continuity on the given function F and the unknown function y. We can usually suppose that F is some simple function with the ordinary properties of continuity, differentiability, etc., as we should almost always be unwise in spending our time in trying to solve the problem in other cases. As, however, most simple functions $F(x, y, p)$ are undefined or discontinuous for certain values of x, y, p, we shall have to be on our guard against the occurrence of such values in our solution, and if they occur satisfy ourselves that I still exists. In any case we have knowledge in advance of the possible abnormalities of F.

With regard to the unknown function y we have no such knowledge; but we may *postulate* that y is to be restricted in some way so as to ensure the existence of I; *e.g.* we may postulate that throughout the range of integration both y and its derivative exist and are continuous, or, more generally, that the range of integration can be divided into a finite number of parts, in each

of which these conditions hold. Using a convenient phrase, which I take from Professor Bolza, let us call any function $y=f(x)$ satisfying these conditions, or the corresponding curve, an *admissible* function or curve. We can replace our original problem by the more restricted problem of finding an *admissible* function y such that for this function I is less than for any other *admissible* function Y. As, however, the conditions thus imposed on y are by no means necessary for the existence of I, the solution (if any) of the restricted problem is not necessarily a solution of the original problem, as some non-admissible function might give a smaller value of I.

By comparison of the integral of $F(x, y, p)$ with that of $F(x, Y, P)$, where Y and its derivative P are in the "neighbourhood" of y, p, *i.e.* differ from y, p respectively by numbers less than an arbitrarily small positive number ϵ, we find by a familiar process that y must satisfy Euler's differential equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial p} = 0$. Even here there is a difficulty, as, except in the very special case when F is linear in p, Euler's equation involves the *second* derivative of y, whose existence has not been postulated. The difficulty can, however, be evaded by alternative methods due to Du Bois Reymond and Professor Hilbert, and Euler's equation can be established without postulating the existence of the second derivative.

If Euler's equation has no admissible integral satisfying the boundary conditions, our restricted problem has no solution; if the equation has one (or more) such integrals, one of them gives the solution of our problem if *it exists*; but we have no proof that such an integral solves our problem. In other words, the condition expressed by Euler's equation is *necessary*, but not *sufficient*.

Legendre in 1786 obtained a second necessary condition $\frac{\partial^2 F}{\partial p^2} \geqslant 0$, for all values of x, y, p occurring in the supposed solution.

The next important step was taken by Jacobi (in 1837), who established a third necessary condition, most simply expressed in geometrical language. In the simplest case, when the end-points A and B are given, a family of "extremals," *i.e.* solutions of Euler's equation, are taken which pass through A; then, if the envelope of this family meets the extremal through A and B in a point C, between A and B (exclusive), the extremal AB is not a solution of the problem, the case when C coincides with B requiring special consideration; and there are corresponding conditions of a slightly more complicated character when the end-points are not given, but restricted in other ways.

Jacobi and his successors up to the time of Weierstrass seem generally to have believed that these three necessary conditions, Euler's, Legendre's and Jacobi's, were also sufficient for the existence of a minimum integral.

Weierstrass's fundamental ideas appear to have been given for the first time systematically in lectures in 1879. They gradually became known through publications by his pupils, and by manuscript copies of lecture notes taken by his hearers, to be found in various libraries. In 1904 Professor Bolza and Professor Hancock each published in America a volume of *Lectures on the Calculus of Variations*, in which free use was made of Weierstrass's lecture notes. From that time onwards the essential features of Weierstrass's work may be regarded as easily accessible, though it is much to be regretted that his actual lectures have not yet been published.

As far as I know * Weierstrass dealt only with a limited part of the subject, and did not discuss cases where the data of the problem involved derivatives of order higher than the first, nor the theory of double integrals. Within these limits he subjected the theory to a critical examination, and apparently for the first time distinguished clearly between necessary and sufficient conditions. He gave a set of necessary conditions, and also a set of sufficient conditions, and, by working systematically with x and y as functions of a parameter, was able to extend largely the class of admissible curves, which in the form $y=f(x)$ were necessarily restricted to curves met in only one point by a line parallel to the axis of y.

His most striking discovery was a fourth necessary condition, which includes

* I have not seen a MS. set of lecture notes.

Legendre's as a special case. Previous writers had apparently assumed tacitly that if a "variation" δy or $Y-y$ is "small," then the variation δp of the derivative is also small, or had at any rate ignored the need for considering variations for which δy is small but δp not small. That a function may be small but its derivative not small, or that the limit of $f(x)$ may be zero and that of $f'(x)$ not zero, is a commonplace of the modern theory of limits.* If, therefore, we have proved that the integral of $F(x, y, p)$ is less than that of $F(x, Y, P)$, when $Y-y$ and $P-p$ are *both* small, we have no guarantee that this will be the case when $Y-y$ only is restricted to be small, and an examination of variations of this latter type, sometimes called "strong" variations, leads to the Weierstrass condition. For example, if the extremal given by Euler's equation is the straight line $y=0$ joining the end-points A and B, we may take as a neighbouring curve a straight line AC making any acute angle with AB and a straight line CB meeting $y=0$ at a small angle ϵ; the corresponding ordinate Y is clearly an admissible function, and by diminishing ϵ we can ensure that Y shall be arbitrarily small throughout, whereas the derivative P remains finite along AC.

Space prohibits more than the slightest reference to later work. The most interesting results have been perhaps Professor Bolza's discovery of a fifth necessary condition (1906), Professor Hilbert's use of his invariant integral, closely connected with Jacobi's condition (1900), and his *a priori* proof of the existence in certain cases of a minimum integral, the integral itself not being known (1899-1901).

The present position of the subject is still unsatisfactory, even with regard to the simple class of problems to which this article has been limited. Sets of necessary conditions and sets of sufficient conditions can be given in one or other of several slightly different forms, but no one has yet succeeded in producing a single set of conditions that are both necessary and sufficient. Any day some one may discover a 6th, 7th, ... necessary condition independent of those at present known.

Professor Bliss is perhaps the most distinguished of a group of American mathematicians who have contributed valuable results to the Calculus of Variations, and unlike some specialists he has succeeded in writing a very simple, lucid, and attractive though accurate book on the subject. It is pleasantly and simply printed without the complications in notation, perhaps unavoidable in a more advanced book, which give Professor Bolza's *Lectures* and his larger *Variationsrechnung* (1909) a somewhat forbidding appearance.

After giving a short introduction, mainly historical, the author deals in detail with three classical problems: the shortest distance between two points, with an extension to the shortest distance between a point and a curve; the brachistochrone (including the case when either or both of the end points is free to move on a given curve); and the surface of revolution of minimum area generated by an arc joining two fixed points, with applications to soap-bubble problems. In each case the problem is completely solved, save for one or two rather tiresome exceptional cases, in which the result is given without proof, and the general principles of the subject are gradually introduced in connection with these problems. Jacobi's condition, established by means of a special case of the Hilbert invariant integral, occurs quite early, in the problem of the shortest distance from a point to a curve.

A final chapter treats of the general integral $\int F(x, y, p)dx$, with fixed or variable end-points, and establishes, in addition to the Weierstrass necessary condition, a set of sufficient conditions for the existence of a minimum.

A feature of the book is the systematic use of the Hilbert invariant integral; Weierstrass's parametric method occurs more rarely.

It may seem ungracious to complain of the author of a book that he has not written instead a different book. But I hope that Professor Bliss will forgive me if I express a hope that when a second edition appears the author will add a treatment on the same lines of the simplest class of so-called isoperimetric problems, in which one integral is to be made a minimum subject to

* The edge of a fret-saw is a gross illustration, which I have often found illuminating.

the condition that a second integral is constant. Some of these problems, *e.g.* that of a closed curve of given length and maximum area, are so interesting that a book, however elementary, which omits them leaves a sense of incompleteness.

I permit myself a few criticisms in detail. The author is presumably writing primarily for American students, among whom there appears to be a tradition of studying advanced subjects without any extensive knowledge of more elementary subjects. It is also the avowed object of the series in which the book appears to provide for readers of this class, and the author has accordingly assumed very little knowledge of general analysis. An English student, on the other hand, would seldom attempt any serious study of such a subject as the Calculus of Variations without a good deal of experience in the manipulation of ordinary functions and a fair general knowledge of analysis, so that he would find, *e.g.*, the author's somewhat apologetic introduction of hyperbolic functions quite unnecessary and perhaps a little ridiculous. A more serious criticism on the same lines is that the introduction of the Hilbert invariant integral strikes me as rather tedious and very artificial, whereas the use of Beltrami's discovery that the left-hand side of Euler's equation can be expressed in the form $\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}$, and a familiar transformation of a double integral, leads naturally and immediately to the invariant integral and its principal properties. This method, though involving more advanced analysis than that of the text, might with advantage be given at least as an alternative.

The interesting historical references would be more useful if in all cases made explicit, whereas in some cases the reader is merely sent to a larger book or encyclopaedia article, where the reference can presumably be found; and the two lists of references would be simpler to use if they were more clearly distinguished, *e.g.* by the use of Roman numerals in one. I should also welcome references to the *Cours d'Analyse* of Jordan and Vallée Poussin, in addition to that of Goursat.

Figures 16 and 21 are badly drawn, as they do not represent the cycloids as cutting the line of cusps at right angles; and I have noticed a single trivial misprint, the date 1844 instead of 1744, in note 7 of p. 182. Otherwise I have detected no mistakes.

King's College, Cambridge, ARTHUR BERRY.
13th Oct., 1925.

Orders of Infinity. The 'Infinitärcalcül' of Paul du Bois-Reymond. By G. H. HARDY. Second edition. Pp. viii, 78. 6s. 1924. (Cambridge Tracts, No. 12, C.U.P.)

The missionary work of this tract was accomplished by the first edition. Everyone is familiar now with $\succ$ and O, and there is even risk in the disseminated knowledge that higher mathematics uses the very language which comes naturally to the schoolboy whose ideas on tendencies and limits are still amorphous; no one can acquire a knack of using epsilons in a manner both plausible and worthless, but teachers will have to be alert to know when a verbal argument corresponds to a real grasp of the subject. This is not an imaginary danger. In his last book, after disavowing any intention of introducing double limits, Dr. Leathem offers a proof of an approximation to the sine which depends on the assertion that if each term in a convergent sequence of functions of a variable is of order higher than the third, the limit must be of order higher than the third; this assertion is disproved by such an example as $nx^4/(1+nx)$, and it is incredible that if the argument had been conducted in symbols a mathematician of experience would have supposed that it did not involve a double limit or that it was valid.

Save that the proof that any function finitely definable by means of logarithms and exponentials is ultimately continuous and monotonic has been recast completely, the text of the seven chapters of the original tract is reproduced with comparatively little modification in the first five chapters of this edition, which contain therefore an account of scales of infinity in general and of logarithmico-exponential scales in particular. A few important

Tauberian theorems for inferring the order of magnitude of a derivative are inserted, a change from "λ assumes values less than any assignable positive number" to "the lower bound of λ is zero" reflects an improvement in the teaching of analysis since 1910, and the reduction in the number of chapters will make references to one edition of the tract impossible to identify from the other, but the substantial alteration is in the account of applications.

Instead of a collection of results in an appendix, we have a chapter, one-third of the whole tract, which the author hopes "may now be useful as an introduction, from a special point of view, to a large field of modern research." To this end, such advice on reading as was on p. 55 of the first edition is, I think, more helpful than the formal references to the unannotated bibliography with which the tract concludes, which are all that Prof. Hardy now gives; such advice represents of course a personal opinion, and is in contrast with the impersonal certainty proper to mathematics, but Prof. Hardy is a teacher whose guidance we covet. Since there is no index to the tract, an analysis of the chapter on applications, in the table of contents or at the head of the chapter itself, would have been welcome. The results given in the old second appendix are repeated or brought up to date—in 1910 it was thought that $\pi(n) - Li(n)$ might be $O(\sqrt{n}/\log n)$, but the difference is now known to be not less than $O(\sqrt{n} \log\log\log n/\log n)$. The additions include: proofs of Stirling's theorem and of a theorem on the increase of an integral function; asymptotic formulae for the partial sum of a divergent series, for oscillating Dirichlet integrals, and for a few arithmetical functions; and accounts of the construction of irregularly increasing functions by means of power series, of the asymptotic behaviour of the real solutions of certain differential equations, and of the classification of irrational numbers.

Enough has been said to shew the value of the new edition to anyone who wishes to work or to read in any branch of modern analysis, and it remains only to deplore that an increase from 70 pages to 86 is accompanied by an increase from half-a-crown to six shillings, the price at which a tract of 114 pages was published in 1920. E. H. N.

Introduction Géométrique à la Mécanique Rationnelle. Par C. Cailler. Pp. xii + 628. 1924. 60 fr. (Georg et Cie.; Gauthier-Villars.)

The late professor of mathematics at Geneva was known abroad for his developments of a theory of loaded figures, and this volume, prepared from the manuscript on which he was working during the last six months of his life, enables us to share the regret felt in his native city at the premature retirement and death of an inspiring teacher.

The first part of the book is a compact account of portions of the algebraic theory of linear and quadratic forms and of linear transformations, in any number of variables; the author, while maintaining that this theory is fundamental to his subject, admits that some readers will benefit most by returning to it as its necessity is borne home. The second part deals with vectors, motors, and line geometry, and concludes with a chapter on the cylindroid. The third part, on finite congruent transformations, direct and perverse, includes an introduction to quaternions, "l'élément analytique indispensable à la théorie des transformations orthogonales à trois et quatre indéterminées." The fourth part carries a theoretical discussion of infinitesimal motion as far as the calculation of the projections of an acceleration in terms of any curvilinear coordinates, and then turns to such practical matters as cams and cogs and Amsler's planimeter.

Not only in the first part but in details throughout M. Cailler is revealed as an analyst at heart. This is illustrated in the quotation above: to Hamilton a quaternion was not an analytical tool, and to operate by means of a frame of reference was a confession of incomplete mastery, but in this book the manipulation of vectors, quaternions, lines in space, or whatever it may be, is almost always by means of coordinates. As an extreme instance of reliance on algebra, the connection of the conjugate of the product of two quaternions with the individual conjugates, which is an immediate logical consequence of the conception of the quaternion as an operator, is deduced from formulae for the elements of the conjugate.

On the one subject of moving axes a narrow view is taken; nothing is said of their use with vectors in general, but the two cases of velocity and acceleration are treated separately, the mystifying 'Coriolis acceleration' being introduced. M. Cailler devotes a chapter to oblique axes, on account of the unexpected importance which the theory of relativity has conferred upon them, and he goes so far as to load the axes and to emphasise the symmetry secured by introducing the contravariant and covariant components, but he does not recognise the loaded frame as a vector frame, and he makes no use whatever of oblique axes in his own work. There are other details that tempt criticism, and my own feeling is that the book would be better if the language were more in intrinsic terms and less in terms of coordinates, but the book as a whole is commendable for the excellence of its scheme, and for the care with which the outlines are filled in. A mass of diverse material is organised into a single structure, partly by insistence on duality—not without an occasional reference to noneuclidean geometry—and partly by systematic use of a device derived from Clifford independently by M'Aulay and Study. Until the study of quaternions is universal, M'Aulay's powerful work can not be popularised, but the idea which M. Cailler develops, which we associate with Study's name, can be presented in comparatively elementary terms, and since as far as I know no English text-book touches on the subject, an attempt to explain the gist of it will not be impertinent.

Intrinsically a motor is a compound of a rotor $\mathbf{r}_l$ and a vector $\mathbf{c}$. The vector $\mathbf{c}$ is parallel to the axis l, but is not located in any particular line; the ratio of $\mathbf{c}$ to $\mathbf{r}$ is a pure number, the pitch of the motor. We think of $\mathbf{c}$ in the first place as the momental vector of some couple, and the motor as the sum of $\mathbf{r}_l$ and the couple, and when we have reached a stage at which we need not distinguish a couple symbolically from its moment we can denote the motor by $\mathbf{r}_l+\mathbf{c}$. If $\mathbf{c}$ is a vector that is not parallel to l, $\mathbf{r}_l+\mathbf{c}$ still denotes a motor, but analysed with reference to the particular line l; it is important to notice that the central axis of $\mathbf{r}_l+\mathbf{c}$ is necessarily a line parallel to l and that the central rotor has the vector $\mathbf{r}$ whatever the vector $\mathbf{c}$ may be. To exhibit the ratio of a motor B to a motor A, or in other words to analyse the operation of converting A into B, Clifford introduces an operator ω which selects from any motor the vector of its rotor part: if A is $\mathbf{r}_l+\mathbf{c}$, then $\omega A=\mathbf{r}$, whether or not l is the central axis. If the vector $\mathbf{r}$ is regarded as the vector part of a degenerate motor whose rotor is nul, the operator ω when applied to $\mathbf{r}$ produces the nul vector; hence $\omega^2 A$ is nul for any motor A, and symbolically $\omega^2=0$. On the other hand, if the rotor $\mathbf{r}_l$ is regarded as the rotor part of a degenerate motor, ω replaces this rotor by its own vector $\mathbf{r}$: the effect of the operator on a vector located in a line is to set it free. The last consideration suggested to Clifford another way of utilising the operator. Given a fixed origin of reference O, any motor can be expressed in the form $\rho+\mathbf{c}$, where ρ is a rotor through O. There is a unique rotor γ which passes through O and has $\mathbf{c}$ for its vector, and by regarding $\mathbf{c}$ as set free from O by the operator ω, we have an expression of the form $\rho+\omega\gamma$ for an arbitrary motor, and we reduce the field of operation of our calculus to consist homogeneously of rotors through a single point. Clifford's next step, though irrelevant to a review of M. Cailler's book, deserves a glance. The interrelations of concurrent rotors are the same as those of their vectors, and it is easy to deduce that ω can consistently be regarded as commutative with any quaternion operator. If $\rho+\omega\gamma$, $\sigma+\omega\delta$ are two motors A, B, and ρ is not nul, σ/ρ is a definite quaternion p, and $pA=\sigma+p\omega\gamma=\sigma+\omega p\gamma=B-\omega(\delta-p\gamma)$, where it must be noticed that in general $p\gamma$ and $\delta-p\gamma$ are quaternions, not rotors; the quotient $(\delta-p\gamma)/\rho$ is a quaternion q, and since no term involving ω^2 is allowed to survive, $\omega qA=\omega(\delta-p\gamma)$, whence $(p+\omega q)A=B$. Thus we have Clifford's conclusion, which is the starting point of M'Aulay's theory of octonions: the ratio of two motors is expressible in the *biquaternion* form $p+\omega q$, unless * the rotor of the denominator is nul.

* The case of exception is inevitable, for a motor which has no definite location in space can not enable us to locate any other motor.

Since, as we have said, the interrelations of concurrent rotors are the same as those of their vectors, we may, if we please, replace $\rho+\omega\gamma$, which is an *analysis* of a motor, by a *description* $\mathbf{r}+\epsilon\mathbf{c}$, from which the motor can be reconstructed when the origin of reference is known; in this substitution, ω, which would annihilate $\mathbf{c}$, must be replaced * by a scalar unit ϵ whose function is to distinguish the parts played by the two vectors, and the law of operation $\omega^2=0$ implies that this unit ϵ satisfies the symbolic law of multiplication $\epsilon^2=0$. When the idea of describing a motor by a complex vector of the form $\mathbf{r}+\epsilon\mathbf{c}$ has been grasped, the nature of the developments of which M. Cailler gives an account can be appreciated at once. The conditions for the motor to reduce to a unit rotor are that the projected product of $\mathbf{r}$ and $\mathbf{c}$ is zero and the projected square of $\mathbf{r}$ is unity, and because $\epsilon^2=0$ these conditions together can be compressed into the one statement that the projected square of $\mathbf{r}+\epsilon\mathbf{c}$ is unity. Hence, by means of this unit ϵ, the geometry of unit rotors, or of directed lines in space, is assimilated to that of unit vectors, or of points on a fixed sphere. Thus for example Frenet's formulae in the theory of a twisted curve can be interpreted without further demonstration as properties of a ruled surface, and if a rigid body is located not by its points but by lines attached to it the body becomes a kind of spherical lamina. A directed line loaded with an ordinary number is a rotor, but if the load is of the form $h+\epsilon k$, where ϵ is the same symbolic unit as before, the result is easily identified with a motor; since a loaded unit vector in real space determines with reference to a fixed origin some point of space, the theory of screws is developed by a revised reading of the elementary theorems of solid geometry.

Is there any reason why students should not be made familiar with a powerful method of this kind at an early stage ? If not, this book should be widely read in our colleges. In any case, it will fascinate anyone who reads mathematics for enjoyment, and it should be in every serious mathematical library.

Congratulations are due to M. Wavre, to whom M. Cailler's manuscript was entrusted; his task included subdividing the material and inserting cross references, and to say that his editorial hand is nowhere to be detected is to pay him very sincerely the compliment that he would desire. E. H. N.

Précis de Mécanique Rationnelle. By G. BOULIGAND. Pp. viii + 282. 25 fr. 1925. (Vuibert.)

In his preface the author states that his chief aim is to accustom students rapidly to the working of dynamical problems. For this purpose such preliminaries as kinematics and the fundamental dynamical principles are treated somewhat briefly, but considerable space is devoted to the analytical method of Lagrange and its application to the continuous and impulsive motion of systems. Eighty pages are occupied by a selection of special problems, worked uniformly by this method. Of the higher dynamical theory, the chief topic treated is the equivalence of any conservative dynamical problem to one of geodesics in a suitably chosen Riemann space, and there is a brief section on the Hamiltonian form of the equations and contact transformations.

The main appeal of the work is thus analytical rather than mechanical. It would be rather " strong meat " for the novice, but those who already have some fair acquaintance with rigid dynamics could derive much profit from its systematic and very general treatment of the subject. The style and language are throughout clear, concise and accurate. Vectorial methods are used whenever convenient, which should enhance the value of the work in a country where they receive at best only a grudging acceptance in the standard textbooks.

Coming now to a more detailed consideration of the subject matter, the first two chapters contain a sketch of the portions of vector-theory and of kinematics relevant to what follows; the treatment of the motions of a rigid

* M'Aulay, though he differs from Clifford in distinguishing the quotient of two concurrent rotors from the quotient of the corresponding vectors, asserts that his scalar unit *is* Clifford's operator under another name.

body is particularly elegant. Then follows an exposition of the fundamental principles and theorems of dynamics. Due prominence is given to the part played by the reference system in the formulation of these principles, and to the (Newtonian) Principle of Relativity, but hardly enough stress is laid on the physical basis of the concepts of force and mass.

In Chapter V. we enter on the main theme of the book, the method of generalised coordinates. By using either Hamilton's Principle or the "general equation of dynamics"

$$\Sigma\{(m\ddot{x}-X)\delta x+(m\ddot{y}-Y)\delta y+(m\ddot{z}-Z)\delta z\}=0,$$

Lagrange's equations are established successively for the cases of a free particle, a particle on a smooth surface, a system of particles, and finally for non-holonomic systems. Some needless repetition is incurred by treating all these cases separately. Special attention is paid to the question of constraints and to the calculation of the forces of constraint.

The next chapter deals amongst other things with the readily integrable cases, which lead to equations of the form

$$x^2=f(x),$$

and it is shown how, by an inspection of the graph of $f(x)$, the qualitative character of the motion may be determined ; this theory deserves to be more widely known than an inspection of English text-books would lead one to suppose. The method is applied throughout the worked problems which compose Chapter VIII. Examiners may note the advice on p. 158 : "On choisit k paramètres $q_1, q_2, \ldots q_k$ susceptibles de déterminer sans ambiguité l'état du système. Si le professeur est un sage, il impose lui-même ses paramètres, pour s'épargner d'inutiles soucis de correction." The last chapter is notable for its discussion, following Hadamard, of geodesics on surfaces in non-Euclidean space.

The author promises a second volume on friction and the dynamics of continuous media. T. M. C.

Nouvelles Vues Faraday-Maxwelliennes. By C. L. R. E. Menges. Pp. 43. 8 fr. 1924. (Gauthier-Villars.)

This book, which is a supplement to a previous publication on the same subject, deals with a particular theory belonging to the category of so-called "emission theories." They originate in an idea put forward by W. Ritz in 1908 in order to give an alternative explanation of the Michelson-Morley experiment. The characteristic feature of this class of theories is that the velocity of the light *relative* to the source is a universal constant c, so that if the source has a velocity v, the absolute velocity of the light leaving the moving source is $c+v$.

Various modifications have to be made in dealing with the reflection of light from a moving mirror, and the experiments of Michelson and Majorana have been used in order to decide between the various modifications which have been proposed.

One of the great difficulties that an emission theory has to face is that of obtaining the value of the Fresuel coefficient in the well-known Fizeau experiment. The type of theory put forward by the author of this book does not appear to be any more successful in this respect than any of the other theories. It rests on an artificial assumption in regard to the meaning to be attached to the index of refraction of the moving medium from which the author attempts to derive mathematically the necessary equations. This deduction as given in *Comptes Rendus*, vol. 175, p. 574 appears to me to be quite unsound as a piece of mathematical reasoning.

The experiments of Michelson and Majorana are also discussed in the light of the author's theory. One conclusion on the last page of the book could easily be tested from a knowledge of the dimensions of Majorana's apparatus. I cannot imagine why the author has failed to obtain this information, which would enable the whole matter to be settled once and for all. H. R. Hassé.

1. **Elementary Geometry.** By C. V. DURELL. Pp. x + 300 + Index 2 + Answers 10. 4s. 6d. 1925. (Bell.)

2. **A Shorter Geometry.** By W. G. BORCHARDT and REV. A. D. PERROTT. Pp. xi + 258 + Answers xi. 4s. 1925. (Bell.)

Mr. Durell adopts the sequence and recommendations set out in the 1923 Report of the I.A.A.M., and arranges the work in three stages—A, Fundamental Concepts, 16 pp.; B, Fundamental Facts, 36 pp.; C, Deductive Development, 219 pp. The remainder of the book is occupied by Appendices, dealing with (I.) Proofs of Fundamental Theorems omitted from the text, (II.) Some very interesting notes on Limit Methods of proof, (III.) Concurrency Properties of Triangles. The "facts" stated in Stage B are Euclid I. 13, 14, 15, 27, 28, 29, 30, 32, 4, 26, 8, together with a preliminary treatment of Similar Triangles based on the idea of Drawing to Scale, also Tangent and Cosine of an Acute Angle followed by a three-figure table of these trigonometrical functions. Of the propositions, number 32 is proved. The others are stated after exercises appealing to intuition.

Messrs. Borchardt and Perrott's book is founded on the authors' *Geometry for Schools*, but the subject-matter has been entirely rearranged to comply with the recommendations of the I.A.A.M. Report. The first few pages, dealing with Fundamental Concepts, are followed by instructions as to measurement, and the simple geometrical constructions are then given *without proof*. Then the Fundamental Facts are generally verified by geometrical drawing and measurement. These are Euclid I. 13, 14, 15, 28, 27, 29 (with proof by Playfair's Axiom), 30 (with proof), 32 (with proof), 4, 26, 5 and 6 (with proofs), 8, the right-angled triangle congruence (with proof), the ambiguous case, and a statement as to similar triangles. This is followed by proofs of the simple geometrical constructions and the general deductive development.

It is well for the I.A.A.M. Report that such able exponents have adopted it. Both books proceed with the Deductive Development of the subject in the order: Triangles and Parallelograms, Intercept Theorems, Areas of Triangles and Parallelograms, Pythagoras' Theorem and its Converse, Circles, Areas of Rectangles and Extension of Pythagoras' Theorem, Rectangle Properties of Circles, and finally Ratio and Similar Figures.

In Mr. Durell's book the two Fundamental Loci follow Pythagoras' Theorem and its Converse, while Messrs. Borchardt and Perrott introduce them somewhat earlier.

Many text-books in common use, including this by Messrs. Borchardt and Perrott, give incomplete proofs of the converse theorems dealing with the angle-conditions which make four points concyclic. Mr. Durell's proofs of these theorems, on the contrary, will be found convincing. Both books are well illustrated with numerous examples and admirably printed. A very conspicuous feature of Mr. Durell's book is the great abundance and variety of the exercises. W. J. DOBBS.

The Earth: Its Origin, History and Physical Constitution. By HAROLD JEFFREYS. Pp. ix + 278. 16s. 1925. (Cambridge University Press.)

Geophysics is one of the few branches of science that have suffered through an insufficiency of text-book expositions. Since the publication of Osmond Fisher's *Physics of the Earth's Crust* in 1889 no English treatise has been written on quantitative geophysics as a distinct science. Particular aspects have been dealt with; for instance, a number of elastic problems have been worked out in detail by Prof. Love in his Adams Prize Essay, *Some Problems in Geodynamics:* but for the most part, geophysical researches have remained scattered through numerous scientific publications, and no attempt has been made to bring them into their proper relationship. It has been evident for some time past that this pioneer work would have to be attempted, and it is difficult to suggest anyone better qualified for the task than Dr. Jeffreys, to whom so much of the recent progress in geophysics is due.

On the purely qualitative side much has been said, and is still being said,

that can easily be disproved by elementary quantitative considerations. What is termed by Dr. Jeffreys "uncontrolled speculation" is all too common, and hypotheses die hard, especially when associated with the name of some giant of the age, even though he may have thrown them out merely as vague suggestions. We are therefore deeply indebted to the author for guiding our way through the bewildering litter of discarded theories, and more so for presenting to us the data of the subject fashioned into a unified science. Even more important still is the setting-out of the relations of geophysics to cosmogony; so skilfully is this task performed that one scarcely realises the transition in six chapters from one science to the other.

A wide knowledge of several different subjects is necessary in order that the different aspects of geophysics may be harmonised; thus, we find that variations in gravity from place to place are bound up with such questions as the conduction of heat and the behaviour of solids under very great stresses. In perusing Dr. Jeffreys' deductions one has the impression that occasionally the "lines of communication" are thin; at the same time one feels how much credit is due to the author for realising that these long chains of argument exist at all, and one cannot fail to appreciate the elegance of many of the lines of reasoning, which are reminiscent at times of the subtleties of a "Sherlock Holmes" story. It is from one of these stories, indeed, that Dr. Jeffreys chooses one of the ingenious mottoes that head the chapters and appendices of his book.

The opening chapter gives an account of Laplace's Nebular Hypothesis, and it is shown that, apart from the impossibility of the generation of planets by the contraction of a rotating nebula, the angular momentum criterion of Babinet raises a fatal objection to the theory. Accordingly, in the next chapter an exposition is given of the only theory that has so far survived a quantitative examination. From a modification of the Planetesimal Hypothesis of Chamberlin and Moulton a development is suggested that resembles closely the Jeans Hypothesis: the close approach of a passing star raised two protuberances on opposite sides of the sun, the larger one being directed towards the star, and from the break up of this elongated filament the planets were formed, while the star's attraction on the filament accounts for the revolution of the planets about the sun. The subsequent condensation is discussed in detail, and the uncondensed portions of the erupted matter are held to constitute the resisting medium.

The difficulty of accounting for the moon, either by capture or by tidal disruption of the earth near the perihelion of its very eccentric primitive orbit, necessitates a chapter on the Resonance Theory, in which Dr. Jeffreys shows that when heterogeneity of density is taken into account, the coincidence of the periods of the semi-diurnal solar tide and the period of free oscillation of the fluid globe would suffice to raise a protuberance, which becoming detached would form the primitive moon.

The resisting medium has long been the scapegoat of cosmogony; for want of a precise formulation of the types of medium that could persist long enough to produce the required reduction of planetary eccentricities many difficulties that have arisen have been disposed of by a vague reference to some resisting medium. It is, therefore, a great achievement that Dr. Jeffreys, after discussing what kind of medium would really exist shortly after the disruption, should so trace the evolution of this medium that the present eccentricity of Mercury's orbit leads incidentally to an estimate of the age of the solar system that is of the same order of magnitude as other determinations. Such a discussion leads naturally to a review of methods of finding the age of the earth. By several independent lines of argument that show good agreement the age is estimated at between $1{\cdot}3 \times 10^9$ and 8×10^9 years.

By the time we reach Chapter VI. we realise that we are on solid ground. The discovery of radioactivity completely invalidated the Kelvin theory of the cooling of the earth, and the problem has been investigated anew by Dr. Jeffreys with the aid of data collected by Holmes. The inferences drawn, that cooling below 700 km. is insignificant and that radioactivity must fall off with depth, are ready to be tested in the light of matters discussed in later chapters. This chapter is furthermore important, inasmuch as it bridges the

time-gap between the formation of the planets and phenomena observable at the present day.

The next brief chapter is a derivation of the equations of motion of an elastic solid under a state of initial stress equivalent to a hydrostatic pressure. Only those who have actually faced problems of this type can appreciate the full pungency of the quotation from Dante's *Inferno* that has been chosen for this chapter: "Lasciate ogni speranza, voi ch' entrate." Chapter VIII. is devoted to a generalisation of Darwin's treatment of the stresses within the earth that arise from a series of parallel mountain chains.

The treatment of isostasy (Chapter IX.) is prefaced by a critical discussion of the definitions of solids and liquids, and an important provisional classification of vitreous bodies into liquevitreous and durovitreous substances is proposed. The subject is presented in a particularly lucid manner and is linked on to the results of Chapter VI., as also is Chapter X. on the thermal contraction theory of mountain building. In dealing with various other surface features of both earth and moon, Dr. Jeffreys employs in the next chapter qualitative arguments to a far greater extent than in other chapters, or, indeed, than is usual in his researches; one feels, however, that such a course is not followed so much from choice as from necessity.

Of the chapter on Seismology no more need be said than that only those aspects are treated that throw light on the composition of the earth's interior. The classical researches of Clairaut, Callandreau and Darwin on the figure of the earth are summarised in Chapter XIII., and a discussion is given of different laws of density. It is particularly to be noted that the law of Laplace has no claim to recognition save that it makes the corresponding differential equation integrable in finite terms. The chapter concludes with a summary of Dr. Jeffreys' argument that the moon owes its present shape to solidification when much nearer the earth than at present, and that the strength of the interior is sufficient to withstand adjustment to the hydrostatic state.

One of the author's greatest achievements has been to reconcile the observed secular acceleration of the moon's mean motion with known sources of energy dissipation (Chapter XIV.). It was found by G. J. Taylor that about one-fiftieth of the energy is being dissipated in the Irish Sea; subsequently Dr. Jeffreys discovered that tidal friction in the shallow seas of the world, notably the Bering Sea, is sufficient to account for at least eighty per cent. of the dissipation, and may account for practically all of it.

The variation of latitude is discussed in Chapter XV., and certain subjects are for convenience relegated to five appendices.

From the purely mathematical standpoint there are several matters of considerable interest. Analysis has in several places been shortened by the use of the operational methods of Heaviside and Bromwich; these methods are not so widely known as they should be, chiefly because Heaviside's works are not easy to read, and are likely to be studied only by the specialist in electromagnetic theory.

A feature of the quantitative treatment is the frequent use of the argument by orders of magnitude, which may be regarded as occupying an intermediate place between an exact solution and purely dimensional reasoning. Dr. Jeffreys explains fully in his preface that such an argument is often no more inaccurate than many of the classical solutions, which of necessity relate only to idealised cases, and are therefore virtually no more than an indication of the order of magnitude. Since, as a matter of experience, "incorrect geophysical hypotheses usually fail by extremely large margins," the method provides a valuable means of discriminating among a number of suggested causes of a given phenomenon. For instance, from time to time it has been suggested that continents are dragged across the earth's surface, as in the Taylor-Wegener hypothesis, and various forces have been proposed, such as tidal friction, that could bring about this result. An examination of the magnitudes of such forces shows them to amount to only a small fraction of that required to overcome the finite strength of the weakest part of the earth's crust.

Appendix E, on Empirical Periodicities, is a valuable caution to those who are apt to indulge recklessly in Fourier analysis. If, for example, $2n+1$

observations of a certain variable are given for equidistant intervals of the time, it is possible to construct a sum of $2n+1$ harmonic terms (consisting of a constant, n sine terms and n cosine terms) that represents the observations exactly at the $2n+1$ instants, and this statement holds for any set of random observations. When there is *a priori* ground for believing that a term of a certain period should be present, it is customary to identify the corresponding term as representing the effect of the cause postulated. Dr. Jeffreys works out the probable amplitude of the terms obtained by analysing in this way a random set of observations, and obtains a criterion analogous to the well-known Schuster criterion. Thus, if the amplitude of a given term is not greater than this expected value, it would be extremely risky to infer that the known or supposed cause was in fact responsible for the term obtained in the Fourier analysis. Still more pointless would it be to take any term in the analysis and seek for a periodic phenomenon that would give rise to such a term, unless it should happen that the amplitude found was largely in excess of that to be expected from a random distribution.

The late Prof. Knott's determination by a brachistochronic method of the velocities of earthquake waves within the earth is of some interest to the pure mathematician, since it depends on the solution of an integral equation that was discussed by Bateman and Herglotz. Those readers who are interested in the theory of probability will find a subtle little problem discussed in Appendix B, where it is shown that there is a very strong probability that the Tidal Theory of the origin of the Solar System is true.

There is little to criticise adversely in this book. The author's relentless self-criticism seems to have usurped one of the time-honoured functions of the reviewer, so much so, indeed, that scarcely any of the subject matter could be omitted without damaging the presentation of the subject as a whole. As Dr. Jeffreys points out in his preface, much of the work of earlier geophysicists has been of necessity omitted. At the same time, it would have been helpful if he had seen his way to include a chapter on earth-tides, which are intimately related to many of the problems discussed, and which afford some information concerning the interior of the earth. Possibly the author considers the ground sufficiently covered by Love's *Geodynamics*; none the less, it does not seem altogether justifiable to omit estimates, such as that of Lord Kelvin, of the rigidity of the earth from its tidal yielding merely on the ground that more " detailed and definite information can now be derived from seismology." The fact that the periods of the acting forces are widely different in the two cases may effect the result; if the same result is obtained, a comparison of the two methods would add to the coherence of the subject as a whole. Since this question follows naturally on the discussion of the free period of the Variation of Latitude, perhaps Dr. Jeffreys will consider the possibility of adding a further chapter when the need arises for a second edition.

R. STONELEY.

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335. It is only from the various essays of experimental industry, and the vague excursions of minds sent out upon discovery, that any advancement of knowledge can be expected.—Rambler, No. 180, p. 233, Vol. 6 (Murphy's edition).

336. L'infinité de la somme de la progression 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, etc., suit nécessairement d'une propriété connue de l'hyperbole entre les asymptotes, sçavoir, que l'aire comprise entre la courbe et l'asymptote, est plus grande qu'aucune aire finie, ou qu'elle est, en langage vulgaire, infinie.—Ozanam, *Réc. Math.*, vol. i. p. 85, f.n. [Transcribed from ed. of 1790, per Prof. E. H. Neville.]

337. I committed one terrible error at which I grieve very sincerely; but how could I suppose that a professor of mathematics would condescend to review a novel? However, it so happened that I attacked the criticism in the *Edinburgh Review* on *Corinne*, and on my right hand sat the author [of the critique], poor Playfair!—Maria Josepha Lady Stanley to her sister June 6, 1809.

Obituary.

Charles Tweedie, M.A., B.Sc., F.R.S.E.

The death has occurred in Edinburgh, in September, after a long illness, of Charles Tweedie. Readers of the *Gazette* will remember him as the author of two brilliant articles in its pages, the one "A Study of the Life and Writings of Colin Maclaurin" and the other the "Life of James Stirling, the Venetian."

Mr. Tweedie was born in 1868. He was a member of a Scottish Border family, and was educated in Edinburgh. He became a student of Edinburgh University in the days of Tait and Chrystal. In 1890 he graduated M.A. with First-Class Honours in Mathematics and Natural Philosophy. In the same year he took the degree B.Sc. After this he continued his mathematical studies at the Universities of Göttingen and Berlin. He then returned to Edinburgh University, where he became the Lecturer in Pure Mathematics under Professor Chrystal. He held this appointment for over twenty years.

Mr. Tweedie was a man of wide attainments. He was a good linguist, and was well read in the works of the geometers of Britain, France, Italy, and Germany. He wrote a large number of original papers. He contributed a considerable number of articles to the *Proceedings* of the Edinburgh Mathematical Society. Of this Society he was a Past-President. He was the author of certain papers in the *Proc. R. S. Edinburgh*, and he produced many mathematical articles for an encyclopedia. He was keenly interested in the school teaching of mathematics, and, in conjunction with a friend, wrote a well-known *Trigonometry*. For a long time he was a University Inspector of Schools under the Scottish Education Department.

Owing to failing health he was obliged to resign his academic duties some years ago, but was able, in the face of overwhelming difficulties, due to illness, to pursue a long-cherished scheme of writing on the Scottish mathematicians—particularly Maclaurin and Stirling. Some of these papers appeared in vols. viii., ix. and x. of the *Gazette*—two have been mentioned already—others were published in the *Proceedings* of the Royal Society of Edinburgh, and in the *Proceedings* of the Edinburgh Mathematical Society. In continuation of this line of research he published in 1922 a volume entitled *James Stirling, a Sketch of his Life and Works, along with his Scientific Correspondence.* His last publication dealt with Gray, the arithmetician, the Scottish Cocker.

E. M. Horsburgh.

NOTICE.

By permission of the *Mathematical Association*, the article on "*Mathematics and Eternity*," by Miss H. P. Hudson, which appeared in the *Gazette* for January, 1925, is being republished in pamphlet form, and can be obtained from the Student Christian Movement, 32 Russell Square, London, W.C. 1, price 2d.

THE LIBRARY.

160 Castle Hill, Reading.

The Librarian reports the following gifts:

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From Major **H. A. Harman**:

E. Cocker	Arithmetick {53: G. Fisher} - - - - - -	1750
	Cocker's *Arithmetick* was "perused and published by the author's correct copy" by John Hawkins in 1677, after Cocker's death. De Morgan conjectured that the work was Hawkins' own, but the *D.N.B.* thinks subsequent evidence unfavourable to this hypothesis.	

From the Misses **Hudson**:

Reports of the British Association for the following years: 1885, '8; 1901, '2, '7; 1910, '2, '3

From Prof. **E. H. Neville**:

M. Bôcher	Integral Equations - - - (Cambridge 10)	1909
A. Henderson	The Twenty-seven Lines upon the Cubic Surface (Cambridge 13)	1911

The following have been bought:

R. Böger	Ebene Geometrie der Lage - - (Schubert 7)	1900
F. Bohnert	Elementare Stereometrie - - (Schubert 4)	1902
R. D. Carmichael	Theory of Numbers - (Merriman-Woodward 13)	1914
L. E. Dickson	Algebraic Invariants - (Merriman-Woodward 14)	1914
D. Katz	*Psychologie und mathematischer Unterricht* - -	1913
	C.I.E.M. 32. Completing for us Bd. 3 of the German section.	
M. Merriman	*A List of writings relating to the method of least squares, with historical and critical notes* - -	1877
	For purposes of reference.	
K. T. Reye	Geometrie der Lage (2 vols.) {2 (1876, 1879) rep.} - -	1882
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Sydney University Reprints. Series XI. 1924. The Cooling of a Solid Sphere with a Concentric Core of Different Material. By H. S. CARSLAW.

Indivisibles and "Ghosts of Departed Quantities" in the History of Mathematics. By F. CAJORI. Pp. 301-306. (Reprint from *Scientia*, May 1925.)

Note on the Lorentz Group. By J. BRILL. Pp. 630-632. (Reprint from *Proc. Cam. Phil. Soc.* xxii. pt. v.)

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The New Matriculation Geometry. By A. G. CRACKNELL and G. F. PERROTT. Pp. x + 303 4s. 6d. 1925. (University Tutorial Press.)

A Practical Treatise on Fourier's Theorem and Harmonic Analysis for Physicists and Engineers. By A. EAGLE. Pp. xiv + 178. 9s. net. 1925. (Longmans, Green.)

The Mechanical Investigations of Leonardo da Vinci. By I. B. HART. Pp. vii + 240. 4$. 1925. (Open Court Co.)

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Puzzle Papers in Arithmetic. By F. C. BOON Pp. 55. 1s. 6d. 1925. (Mills & Boon.)

Primer of Arithmetic for Middle Forms. By F. M. MARZIALS and N. K. BARBER. Pp. xii + 262. 3s. 6d. net. 1925. (Oxford Univ. Press.)

Exercises in Algebra, from the Beginnings to the Quadratic. By R. W. M. GIBBS. Pp. 160. 1s. 6d. net. 1925. (Oxford Univ. Press.)

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The Theory of Measurements. By L. TUTTLE and J. SATTERLEY. Pp. xi + 334. 12s. 6d. net. 1925. (Longmans, Green.)

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